Oxford Cambridge and RSA

Wednesday 14 October 2020 – Afternoon

A Level Mathematics A

H240/02 Pure Mathematics and Statistics

Time allowed: 2 hours

You must have:
- the Printed Answer Booklet
- a scientific or graphical calculator

INSTRUCTIONS

- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION

- The total mark for this paper is **100**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **12** pages.

ADVICE

- Read each question carefully before you start your answer.

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Formulae
A Level Mathematics A (H240)

Arithmetic series

$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a+(n-1)d\}$

Geometric series

$S_n = \dfrac{a(1-r^n)}{1-r}$

$S_\infty = \dfrac{a}{1-r}$ for $|r| < 1$

Binomial series

$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N})$,

where ${}^nC_r = {}_nC_r = \dbinom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots + \dfrac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1, n \in \mathbb{R})$

Differentiation

$f(x)$	$f'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

Quotient rule $y = \dfrac{u}{v}$, $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Differentiation from first principles

$f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

Integration

$\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

$\displaystyle\int f'(x)(f(x))^n\,dx = \frac{1}{n+1}(f(x))^{n+1} + c$

Integration by parts $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k+\tfrac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\int_a^b y\,dx \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $f(x) = 0$: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Probability

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A)P(B \mid A) = P(B)P(A \mid B)$ **or** $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

Standard deviation

$\sqrt{\dfrac{\Sigma(x-\overline{x})^2}{n}} = \sqrt{\dfrac{\Sigma x^2}{n} - \overline{x}^2}$ **or** $\sqrt{\dfrac{\Sigma f(x-\overline{x})^2}{\Sigma f}} = \sqrt{\dfrac{\Sigma fx^2}{\Sigma f} - \overline{x}^2}$

The binomial distribution

If $X \sim B(n, p)$ then $P(X = x) = \dbinom{n}{x} p^x (1-p)^{n-x}$, mean of X is np, variance of X is $np(1-p)$

Hypothesis test for the mean of a normal distribution

If $X \sim N(\mu, \sigma^2)$ then $\overline{X} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

Percentage points of the normal distribution

If Z has a normal distribution with mean 0 and variance 1 then, for each value of p, the table gives the value of z such that $P(Z \leqslant z) = p$.

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

Kinematics

Motion in a straight line

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u+v)t$

$v^2 = u^2 + 2as$

$s = vt - \frac{1}{2}at^2$

Motion in two dimensions

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$\mathbf{s} = \frac{1}{2}(\mathbf{u}+\mathbf{v})t$

$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

Section A: Pure Mathematics

Answer **all** the questions.

1 **(a)** Differentiate the following with respect to x.

(i) $(2x+3)^7$ **[2]**

(ii) $x^3 \ln x$ **[3]**

(b) Find $\int \cos 5x \, dx$. **[2]**

(c) Find the equation of the curve through (1, 3) for which $\frac{dy}{dx} = 6x - 5$. **[2]**

2 Simplify fully $\frac{2x^3 + x^2 - 7x - 6}{x^2 - x - 2}$. **[4]**

3 In this question you should assume that $-1 < x < 1$.

(a) For the binomial expansion of $(1-x)^{-2}$

(i) find and simplify the first four terms, **[2]**

(ii) write down the term in x^n. **[1]**

(b) Write down the sum to infinity of the series $1 + x + x^2 + x^3 + \dots$. **[1]**

(c) Hence or otherwise find and simplify an expression for $2 + 3x + 4x^2 + 5x^3 + \dots$ in the form $\frac{a-x}{(b-x)^2}$ where a and b are constants to be determined. **[3]**

4 **In this question you must show detailed reasoning.**

Solve the equation $3\sin^4\phi + \sin^2\phi = 4$, for $0 \leqslant \phi < 2\pi$, where ϕ is measured in radians. **[5]**

5 **(a)** Determine the set of values of n for which $\frac{n^2-1}{2}$ and $\frac{n^2+1}{2}$ are positive integers. **[3]**

A 'Pythagorean triple' is a set of three positive integers a, b and c such that $a^2 + b^2 = c^2$.

(b) Prove that, for the set of values of n found in part **(a)**, the numbers n, $\frac{n^2-1}{2}$ and $\frac{n^2+1}{2}$ form a Pythagorean triple. **[2]**

6 Prove that $\sqrt{2}\cos(2\theta + 45°) \equiv \cos^2\theta - 2\sin\theta\cos\theta - \sin^2\theta$, where θ is measured in degrees. **[3]**

7 A and B are fixed points in the x-y plane. The position vectors of A and B are **a** and **b** respectively.

State, with reference to points A and B, the geometrical significance of

(a) the quantity $|\mathbf{a}-\mathbf{b}|$, **[1]**

(b) the vector $\frac{1}{2}(\mathbf{a}+\mathbf{b})$. **[1]**

The circle P is the set of points with position vector **p** in the x-y plane which satisfy

$$\left|\mathbf{p}-\frac{1}{2}(\mathbf{a}+\mathbf{b})\right|=\frac{1}{2}\left|\mathbf{a}-\mathbf{b}\right|.$$

(c) State, in terms of **a** and **b**,

(i) the position vector of the centre of P, **[1]**

(ii) the radius of P. **[1]**

It is now given that $\mathbf{a}=\begin{pmatrix}2\\-1\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}4\\5\end{pmatrix}$ and $\mathbf{p}=\begin{pmatrix}x\\y\end{pmatrix}$.

(d) Find a cartesian equation of P. **[4]**

8 The rate of change of a certain population P at time t is modelled by the equation $\frac{\mathrm{d}P}{\mathrm{d}t}=(100-P)$.

Initially $P=2000$.

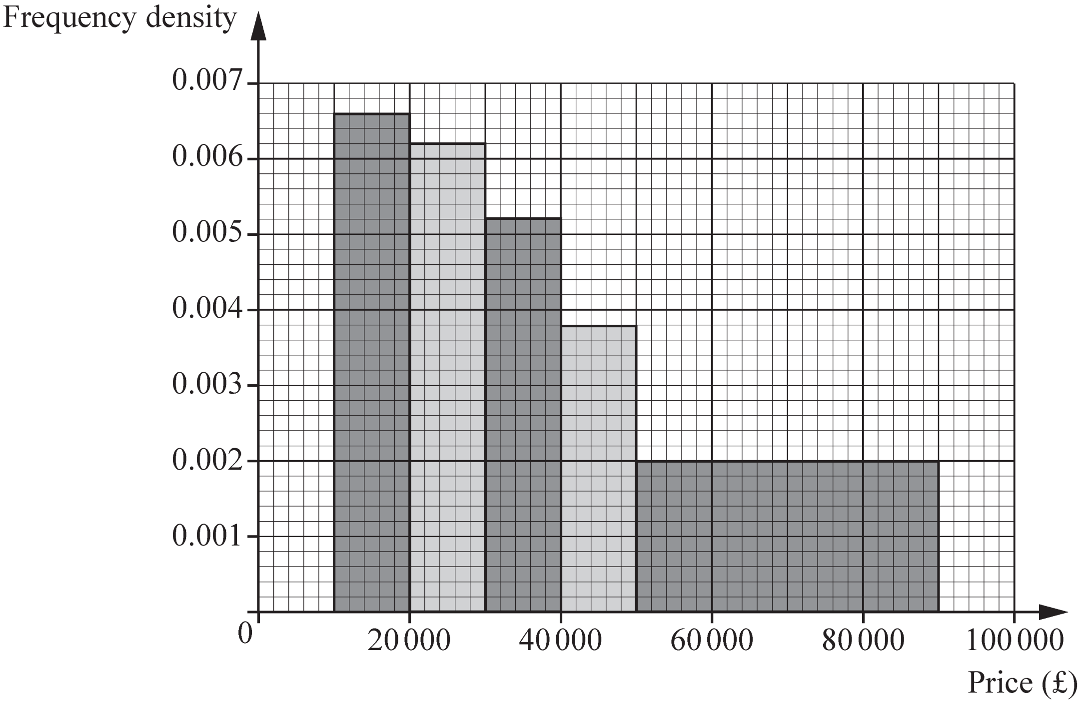
(a) Determine an expression for P in terms of t. **[7]**

(b) Describe how the population changes over time. **[2]**

Section B: Statistics

Answer **all** the questions.

9 The histogram shows information about the numbers of cars in five different price ranges, sold in one year at a car showroom.



It is given that 66 cars in the price range £10 000 to £20 000 were sold.

(a) Find the number of cars sold in the price range £50 000 to £90 000. **[1]**

(b) State the units of the frequency density. **[1]**

(c) Suggest one change that the management could make to the diagram so that it would provide more information. **[1]**

(d) Estimate the number of cars sold in the price range £50 000 to £60 000. **[1]**

10 Pierre is a chef. He claims that 90% of his customers are satisfied with his cooking. Yvette suspects that Pierre is over-confident about the level of satisfaction amongst his customers. She talks to a random sample of 15 of Pierre's customers, and finds that 11 customers say that they are satisfied. She then performs a hypothesis test.

Carry out the test at the 5% significance level. **[7]**

11 As part of a research project, the masses, m grams, of a random sample of 1000 pebbles from a certain beach were recorded. The results are summarised in the table.

Mass (g)	$50 \leqslant m < 150$	$150 \leqslant m < 200$	$200 \leqslant m < 250$	$250 \leqslant m < 350$
Frequency	162	318	355	165

(a) Calculate estimates of the mean and standard deviation of these masses. **[2]**

The masses, x grams, of a random sample of 1000 pebbles on a different beach were also found. It was proposed that the distribution of these masses should be modelled by the random variable $X \sim \mathrm{N}(200, 3600)$.

(b) Use the model to find $\mathrm{P}(150 < X < 210)$. **[1]**

(c) Use the model to determine x_1 such that $\mathrm{P}(160 < X < x_1) = 0.6$, giving your answer correct to **five** significant figures. **[3]**

It was found that the smallest and largest masses of the pebbles in this second sample were 112 g and 288 g respectively.

(d) Use these results to show that the model may not be appropriate. **[1]**

(e) Suggest a different value of a parameter of the model in the light of these results. **[2]**

12 In the past, the time for Jeff's journey to work had mean 45.7 minutes and standard deviation 5.6 minutes. This year he is trying a new route. In order to test whether the new route has reduced his journey time, Jeff finds the mean time for a random sample of 30 journeys using the new route. He carries out a hypothesis test at the 2.5% significance level.

Jeff assumes that, for the new route, the journey time has a normal distribution with standard deviation 5.6 minutes.

(a) State appropriate null and alternative hypotheses for the test. **[2]**

(b) Determine the rejection region for the test. **[4]**

13 Andy and Bev are playing a game.

- The game consists of three points.
- On each point, P(Andy wins) = 0.4 and P(Bev wins) = 0.6.
- If one player wins two consecutive points, then they win the game, otherwise neither player wins.

(a) Determine the probability of the following events.

(i) Andy wins the game. **[2]**

(ii) Neither player wins the game. **[3]**

Andy and Bev now decide to play a match which consists of a series of games.

- In each game, if a player wins the game then they win the match.
- If neither player wins the game then the players play another game.

(b) Determine the probability that Andy wins the match. **[3]**

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Turn over for question 14

14 Table 1 shows the numbers of usual residents in the age range 0 to 4 in 15 Local Authorities (LAs) in 2001 and 2011. The table also shows the increase in the numbers in this age group, and the same increase as a percentage.

	2001	2011	Increase	% Increase
Bolton	16779	18765	1986	11.84%
Bury	11117	12235	1118	10.06%
Knowsley	9454	9121	−333	−3.52%
Liverpool	24840	26099	1259	5.07%
Manchester	24693	36413	11720	47.46%
Oldham	15196	16491	1295	8.52%
Rochdale	13771	14754	983	7.14%
Salford	12529	16255	3726	29.74%
Sefton	14896	14601	−295	−1.98%
St. Helens	10083	10269	186	1.84%
Stockport	16457	17342	885	5.38%
Tameside	12803	14439	1636	12.78%
Trafford	11971	14870	2899	24.22%
Wigan	17561	19681	2120	12.07%
Wirral	17475	18514	1039	5.95%

Table 1

Fig. 2 shows the increase in each LA in raw numbers, and Fig. 3 shows the percentage increase in each LA.

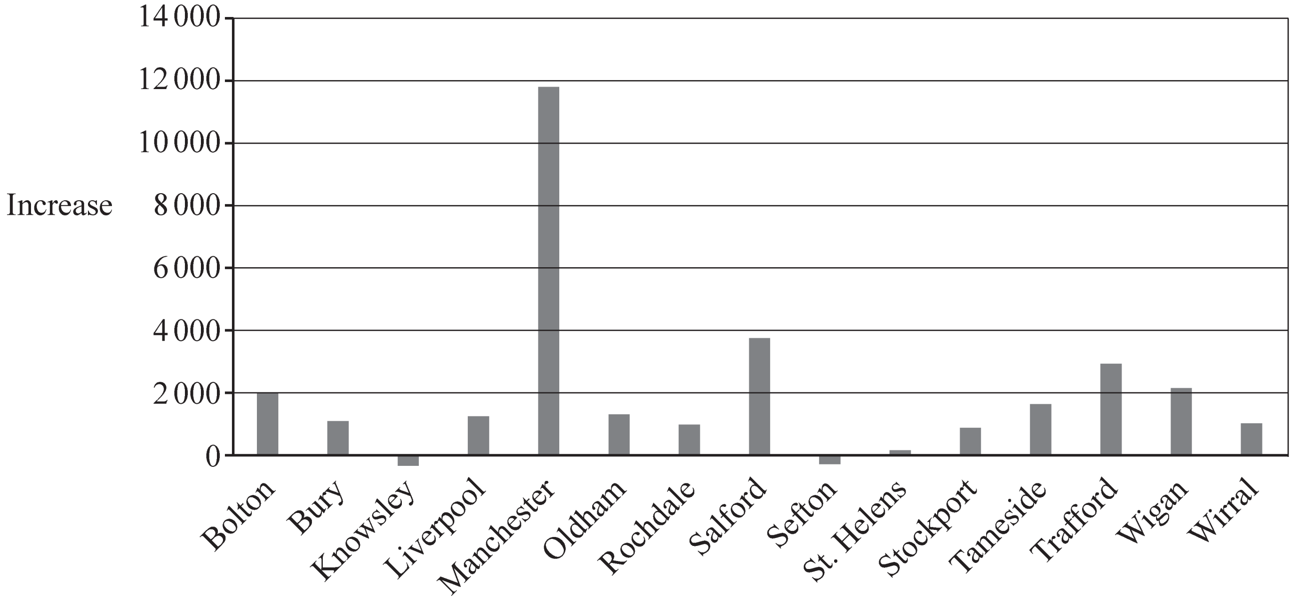

Fig. 2

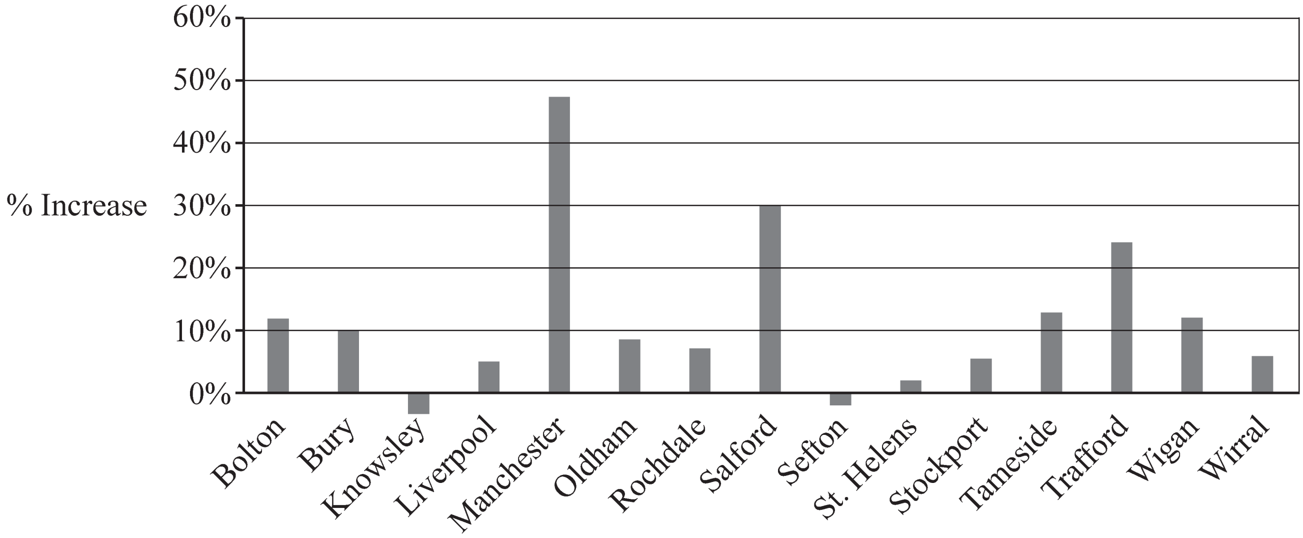

Fig. 3

(a) The Education Committees in these LAs need to plan for the provision of schools for pupils in their districts.

(i) Explain why, in this context, the increase is more important than the actual numbers. **[1]**

(ii) In which of the following LAs was there likely to have been the greatest need for extra teachers in the years following 2011: Bolton, Sefton, Tameside or Wigan?
Give a reason for your answer. **[2]**

(iii) State an assumption about the populations needed to make your answer in part **(ii)** valid. **[1]**

(b) In two of the 15 LAs the proportion of young families is greater than in the other 13 LAs. Suggest, using only data from Fig. 2 and Fig. 3 and/or Table 1, which two LAs these are most likely to be. **[2]**

Turn over for question 15

15 In this question you must show detailed reasoning.

The random variable X has probability distribution defined as follows.

$$P(X = x) = \begin{cases} \frac{15}{64} \times \frac{2^x}{x!} & x = 2, 3, 4, 5, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that $P(X = 2) = \frac{15}{32}$. **[1]**

The values of three independent observations of X are denoted by X_1, X_2 and X_3.

(b) Given that $X_1 + X_2 + X_3 = 9$, determine the probability that at least one of these three values is equal to 2. **[6]**

Freda chooses values of X at random until she has obtained $X = 2$ exactly three times. She then stops.

(c) Determine the probability that she chooses exactly 10 values of X. **[3]**

END OF QUESTION PAPER