Cambridge Assessment Admissions Testing

Test of Mathematics for University Admission

Paper 2 2017 worked answers

Test of Mathematics for University Admission
2017 Paper 2
Worked Solutions

Version 1.0, April 2019

Contents

Introduction for students

These solutions are designed to support you as you prepare to take the Test of Mathematics for University Admission. They are intended to help you understand how to answer the questions, and therefore you are strongly encouraged to **attempt the questions first** before looking at these worked solutions. For this reason, each solution starts on a new page, so that you can avoid looking ahead.

The solutions contain much more detail and explanation than you would need to write in the test itself – after all, the test is multiple choice, so no written solutions are needed, and you may be very fluent at some of the steps spelled out here. Nevertheless, doing too much in your head might lead to making unnecessary mistakes, so a healthy balance is a good target!

There may be alternative ways to correctly answer these questions; these are not meant to be 'definitive' solutions.

The questions themselves are available on the 'Preparing for the test' section on the Admissions Testing website.

Paper 2 uses ideas from mathematical logic. These are explained in detail in the 'Notes on Logic and Proof' on the above webpage.

Question 1

We first expand the brackets and split up the fraction:

$$\begin{aligned} y &= \frac{(1-3x)^2}{2x^{\frac{3}{2}}} \\ &= \frac{1-6x+9x^2}{2x^{\frac{3}{2}}} \\ &= \tfrac{1}{2}x^{-\frac{3}{2}} - 3x^{-\frac{1}{2}} + \tfrac{9}{2}x^{\frac{1}{2}} \end{aligned}$$

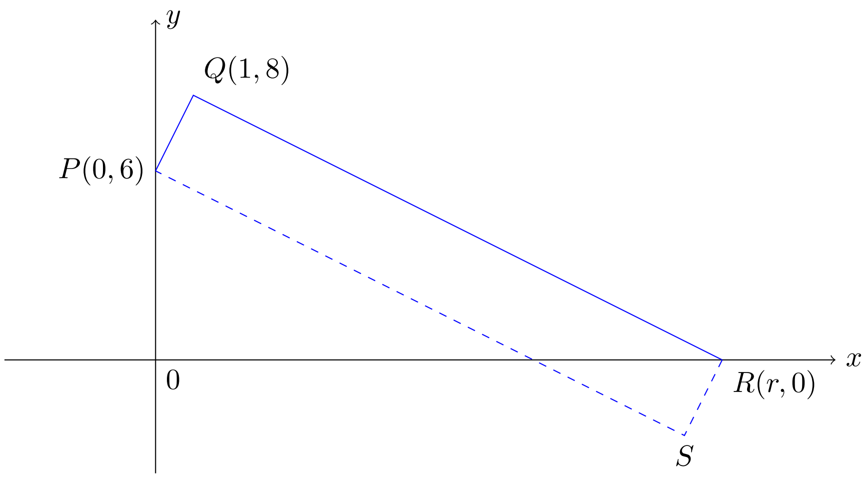
and we can now differentiate this to get

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \tfrac{1}{2}(-\tfrac{3}{2})x^{-\frac{5}{2}} - 3(-\tfrac{1}{2})x^{-\frac{3}{2}} + \tfrac{9}{2} \times \tfrac{1}{2}x^{-\frac{1}{2}} \\ &= -\tfrac{3}{4}x^{-\frac{5}{2}} + \tfrac{3}{2}x^{-\frac{3}{2}} + \tfrac{9}{4}x^{-\frac{1}{2}} \end{aligned}$$

and so the correct answer is A.

Question 2

A quick sketch (not drawn to scale) may help us here.



PQ has length $\sqrt{1^2 + 2^2} = \sqrt{5}$ by Pythagoras's theorem.

The gradient of PQ is $\dfrac{8-6}{1-0} = 2$, so QR has gradient $-\frac{1}{2}$

We could now find the coordinates of R by finding the equation of the line QR and substituting in $y = 0$ to find the x-coordinate of R.

Alternatively, if we set the coordinates of R to be $(r, 0)$, we can just use the known gradient of QR to find r:

$$\frac{0-8}{r-1} = -\frac{1}{2}$$

so $\frac{8}{r-1} = \frac{1}{2}$ giving $r - 1 = 16$, hence $r = 17$

Therefore the length QR is

$$\sqrt{(r-1)^2 + (-8)^2} = \sqrt{16^2 + 8^2} = 8\sqrt{2^2 + 1^2} = 8\sqrt{5}$$

which gives the area of $PQRS$ as $\sqrt{5} \times 8\sqrt{5} = 40$, and the answer is E.

Interestingly, it turns out that we only actually needed $r - 1$; we did not need to know the actual coordinates of R.

Question 3

The first term is $a = 2\sqrt{3}$ and the common ratio is r. So the fourth term is $ar^3 = 2\sqrt{3}\,r^3 = \frac{9}{4}$ and hence

$$r^3 = \frac{9}{8\sqrt{3}} = \frac{9\sqrt{3}}{8 \times 3} = \frac{3\sqrt{3}}{8}$$

by rationalising the denominator.

We can write $3\sqrt{3}$ as $\sqrt{27}$ or $3^{\frac{3}{2}}$ (since we want to take the cube root of this expression), and so we have

$$r^3 = \frac{3^{\frac{3}{2}}}{8} \implies r = \frac{3^{\frac{1}{2}}}{2} = \frac{\sqrt{3}}{2}.$$

(There is a unique real cube root of any real number, unlike the case for square roots.)

We can now find the sum to infinity of the geometric progression:

$$\begin{aligned}
S_\infty &= \frac{a}{1-r} \\
S_\infty &= \frac{2\sqrt{3}}{1-\frac{\sqrt{3}}{2}} \\
&= \frac{4\sqrt{3}}{2-\sqrt{3}} && \text{multiplying by } \frac{2}{2} \\
&= \frac{4\sqrt{3}(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})} && \text{rationalising the denominator} \\
&= \frac{4(2\sqrt{3}+3)}{1} && \text{paying attention to the options offered} \\
&= 4(2\sqrt{3}+3)
\end{aligned}$$

and the correct answer is G.

Question 4

Let's run through every step of the argument with care.

The first step is '$\tan x = \sqrt{3}$ so $x = 60°$'. We should first check that $x = 60°$ is a solution: using our knowledge of trigonometry and special angles, we know that $\tan 60° = \sqrt{3}$, so it is.

But we also know that tan is periodic with period $180°$: there are multiple values of x which have $\tan x = \sqrt{3}$, and all solutions to $\tan x = \sqrt{3}$ are therefore of the form $x = 60° + 180n°$ for $n \in \mathbb{Z}$.

So there is unquestionably an error in the student's answer, but we do not yet know whether there is more than one possible value of $\sin 2x$.

It follows from $x = 60° + 180n°$ that $2x = 120° + 360n°$

It is certainly the case that $\sin 120° = \frac{\sqrt{3}}{2}$. But since sin is periodic with period $360°$, we see that $\sin(120° + 360n°) = \frac{\sqrt{3}}{2}$ for every $n \in \mathbb{Z}$.

So there is only one possible value of $\sin 2x$, but the student should have considered other possible values of x for which $\tan x = \sqrt{3}$; this is option B.

Question 5

We note from the context of the statements that $91 = 7 \times 13$ shows that 91 is not prime. It also shows that 91 is a multiple of 7 and a multiple of 13.

We use a counterexample when we are trying to disprove a statement of the form '**if** A **then** B'; the counterexample will be a case where A is true but B is false. We can also use counterexamples to disprove statements of the form '**for all** x, A is true': a single example of an x for which A is false shows that the statement is false.

We first rewrite each of the statements given into an explicit '**if** . . . **then**' statement or an explicit '**for all**' statement.

(With experience, this is not actually necessary; once you have seen enough statements, it becomes clear what a counterexample 'looks like'. But for the purposes of reaching this point, it can be useful to go through these formal steps.)

1 The word 'when' in this context has the same logical meaning as 'if', so the statement can be rewritten as '**if** p is an odd prime, **then** $10p^2 + 1$ and $10p^2 - 1$ are both prime'.

So if $91 = 7 \times 13$ is to provide a counterexample, we need an odd prime p with $10p^2 + 1 = 91$ or $10p^2 - 1 = 91$, since in that case, '$10p^2 + 1$ and $10p^2 - 1$ are both prime' will be false. It is clear that $p = 3$ achieves this, for then $10p^2 + 1 = 91$.

2 We can write this statement as an explicit '**for all**' statement, as 'Every prime. . . ' suggests this meaning. The statement becomes: '**For all** primes p where $p > 5$, $p = 6n + 1$ for some integer n'.

So a counterexample would be a prime p with $p > 5$ but for which $p = 6n+1$ is false. However, 91 is not prime, so $91 = 7 \times 13$ does not provide a counterexample. The fact that $91 = 6n + 1$ for $n = 15$ is immaterial, and the fact that the statement is false (as $11 = 6 \times 2 - 1$) does not help either.

3 We can again write this as an explicit '**for all**' statement; the English is better with the equivalent '**for each**', though: '**For each** n which is a multiple of 7 greater than 7, n is not prime.'

If we now look at $91 = 7 \times 13$, we see that 91 is a multiple of 7 greater than 7. However, 91 is not prime, so 91 is an example of where the statement does hold, rather than being a counterexample. In fact, the given statement is true, so there cannot be any counterexamples, and we need not have looked at $91 = 7 \times 13$ at all.

So the answer is B.

Question 6

We calculate the first few values of u_n in the hope that this will show a clear pattern.

We have

$$\begin{aligned} u_0 &= 1 \\ u_1 &= \int_0^1 4x.1\,\mathrm{d}x \\ &= \left[2x^2\right]_0^1 \\ &= 2 \\ u_2 &= \int_0^1 4x.2\,\mathrm{d}x \\ &= \left[4x^2\right]_0^1 \\ &= 4 \\ u_3 &= \int_0^1 4x.4\,\mathrm{d}x \\ &= \left[8x^2\right]_0^1 \\ &= 8 \end{aligned}$$

The pattern is now apparent: $u_n = 2^n$, and so the answer is A.

If we wanted to *prove* this to be true, we could use induction. The result is clearly true when $n = 0$, so assume that it is true when $n = k$. Then

$$\begin{aligned} u_{k+1} &= \int_0^1 4x.2^k\,\mathrm{d}x \\ &= \left[2.2^k x^2\right]_0^1 \\ &= 2.2^k \\ &= 2^{k+1} \end{aligned}$$

and so the result holds for $n = k + 1$. Hence by induction, $u_n = 2^n$ for all $n \geqslant 0$.

Question 7

The graph of $y = \mathrm{f}(x)$ looks like an exponential graph, so all of the four given statements certainly seem plausible.

One approach to this question is to think about the shapes of different exponential graphs. Another is to consider a specific value of x, and see what happens.

If we take $x = 1$, $a^x = a$, and $\mathrm{f}(1) > a$. We can now substitute $x = 1$ into the two expressions for $\mathrm{f}(x)$ offered: $\mathrm{f}(x) = b^x$ gives $\mathrm{f}(1) = b$, so we would need $b > a$. Similarly, $\mathrm{f}(x) = a^{kx}$ gives $\mathrm{f}(1) = a^k$. But it is clear from the solid line that $a > 1$ (as $a^0 = 1$ and $a^1 = a > 1$), so $a^k > a$ requires $k > 1$.

So both statements **1** and **3** could be true, but not **2** or **4**.

We can be even more careful and show that we cannot have just one of **1** and **3** being true. If statement **3** is true, we have $\mathrm{f}(x) = a^{kx} = (a^k)^x$, so writing $b = a^k > a$ shows that statement **1** is true; likewise, if statement **1** is true, we have $\mathrm{f}(x) = b^x = (a^{\log_a b})^x = a^{(\log_a b)x}$ and $\log_a b > 1$, so statement **3** is also true.

Hence the correct answer is E.

Question 8

We can start by estimating or calculating the values.

A $2 < \log_2 7 < 3$, since $\log_2 4 = 2$ and $\log_2 8 = 3$

B $(2^{-3} + 2^{-2})^{-1} = (\frac{1}{8} + \frac{1}{4})^{-1} = (\frac{3}{8})^{-1} = \frac{8}{3}$ which again lies between 2 and 3

C $\pi/3$ is just a little greater than 1, so $2^{\pi/3}$ is a little greater than 2

D We expand $(\sqrt{2} - 1)^3$ using the binomial theorem to get

$$(\sqrt{2})^3 - 3(\sqrt{2})^2 + 3\sqrt{2} - 1 = 2\sqrt{2} - 6 + 3\sqrt{2} - 1 = 5\sqrt{2} - 7$$

Thus

$$\frac{1}{4(\sqrt{2}-1)^3} = \frac{1}{4(5\sqrt{2}-7)} = \frac{5\sqrt{2}+7}{4((5\sqrt{2})^2 - 7^2)} = \frac{5\sqrt{2}+7}{4}$$

This is greater than $\frac{5+7}{4} = 3$, so is certainly not the smallest in value.

E $\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$, so $4\sin^2(\frac{\pi}{4}) = 4\left(\frac{1}{\sqrt{2}}\right)^2 = 4 \times \frac{1}{2} = 2$

Since A, B and C are each greater than 2, the correct answer is E.

Question 9

> ***Commentary:*** *This is Fermat's Last Theorem for the case $n = 3$; it was first proven by Euler in the 18th century. The attempted proof here, though, is deeply flawed, as we will see.*

Line I is fine: this is a simple rearrangement.

Line II is fine: this is a standard algebraic identity which is easy to check.

Line III is problematic. It is certainly true that $a \leqslant a^2$ and $c - b < c + b \leqslant c^2 + b^2 < c^2 + cb + b^2$. It is therefore plausible that $a = c - b$ and $a^2 = c^2 + cb + b^2$, but it is far from necessary. Of course we cannot give an explicit example to show this, as there are no sets of positive integers a, b and c with $a^3 + b^3 = c^3$. But we can observe that if $c - b = 1$, we must have $a^3 = c^2 + bc + b^3$, and this possibility has not been considered; alternatively, we could consider a case where a is not prime such as $a = 6$; then we can write $a^3 = 6^3 = 3 \times 72$, and we could have $c - b = 3$ and $c^2 + cb + b^2 = 72$. Either way, it does not necessarily follow that $a = c - b$ and $a^2 = c^2 + cb + b^2$, so this step is wrong.

The correct answer is D.

Question 10

To show that a condition is **sufficient**, we need to show that if the condition is satisfied, then $\int_1^3 f(x)\,dx = 0$

Note that it is not enough for us to find an example of a function which satisfies $\int_1^3 f(x)\,dx = 0$ and then to check whether the function satisfies the suggested conditions, as this would be checking whether the conditions are **necessary**, not whether they are **sufficient**.

A Knowing the value of $f(x)$ at a single value of x cannot hope to force the integral to be 0; consider, for example, the quadratic $f(x) = (x-2)^2$

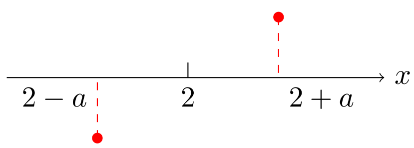
B Likewise, knowing the value of $f(x)$ at the endpoints is not **sufficient**; consider, for example, the quadratic $f(x) = (x-1)(x-3)$

C This says that the function is odd: the values of $f(x)$ for negative values of x match those for positive values of x. But it gives no control over the values of $f(x)$ between $x = 1$ and $x = 3$, so it cannot be a **sufficient** condition. For example, $f(x) = x$ satisfies $f(-x) = -f(x)$, but $\int_1^3 x\,dx > 0$

So it is one of D and E, both of which look complicated. E looks marginally easier to deal with, as $x - 2 = -(2 - x)$. But that means that if we write $y = 2 - x$, condition E becomes $f(-y) = -f(y)$, which is exactly the same as condition C. So condition E is not **sufficient**.

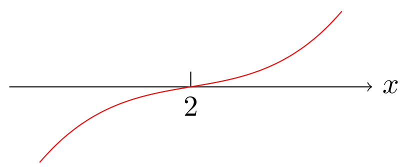
By elimination, therefore, the correct answer must be condition D.

Let us show that D is, indeed, **sufficient**. It would help to see what is going on if we write $x+2$ as $2+x$, so the condition becomes $f(2+x) = -f(2-x)$. If we take $0 < x \leqslant 1$, this shows that there is rotational symmetry about the point $(2, 0)$:



And when $x = 0$, we get $f(2) = -f(2)$ so $f(2) = 0$

So the graph of $y = f(x)$ itself might look like this:



Since the graph of $y = f(x)$ has rotational symmetry about $(2, 0)$, the integrals $\int_1^2 f(x)\,dx$ and $\int_2^3 f(x)\,dx$ will be negative of each other, and so condition D is **sufficient** for $\int_1^3 f(x)\,dx = 0$.

Question 11

We can approach this question algebraically or graphically.

We start with an algebraic approach. We first note that $\mathrm{f}(x) \geqslant 0$ for all $x > 0$ as $\mathrm{f}(x)$ is increasing. So the region described first, with area R, lies entirely above (or on) the x-axis, and

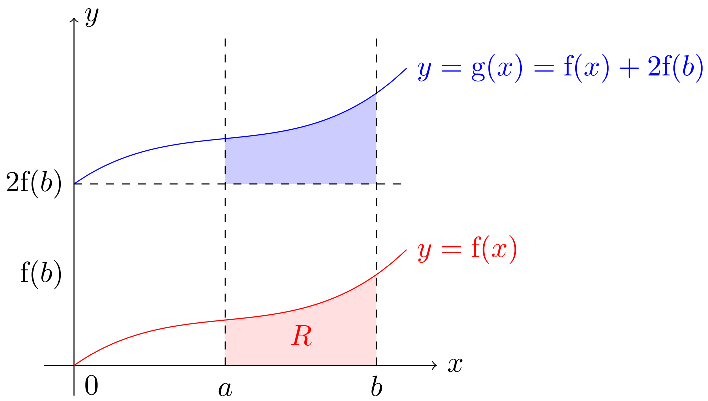
$$R = \int_a^b \mathrm{f}(x)\,\mathrm{d}x$$

We note that $\mathrm{g}(x) = \mathrm{f}(x) + 2\mathrm{f}(b) \geqslant 0$ for $x > 0$ as $\mathrm{f}(x) \geqslant 0$ and $\mathrm{f}(b) \geqslant 0$. Then the area required under the graph $\mathrm{g}(x)$ is given by

$$\begin{aligned}\int_a^b \mathrm{g}(x)\,\mathrm{d}x &= \int_a^b \mathrm{f}(x) + 2\mathrm{f}(b)\,\mathrm{d}x \\ &= \int_a^b \mathrm{f}(x)\,\mathrm{d}x + 2\mathrm{f}(b)\int_a^b 1\,\mathrm{d}x \\ &= R + 2\mathrm{f}(b)(b-a)\end{aligned}$$

On the second line we have split the integral into a sum of two integrals, and taken out the *constant* factor of $2\mathrm{f}(b)$ from the second integral. Thus the answer is option B.

We could also work graphically. We can sketch the graphs of $\mathrm{f}(x)$ and $\mathrm{g}(x)$, making up a suitable increasing shape for $\mathrm{f}(x)$:



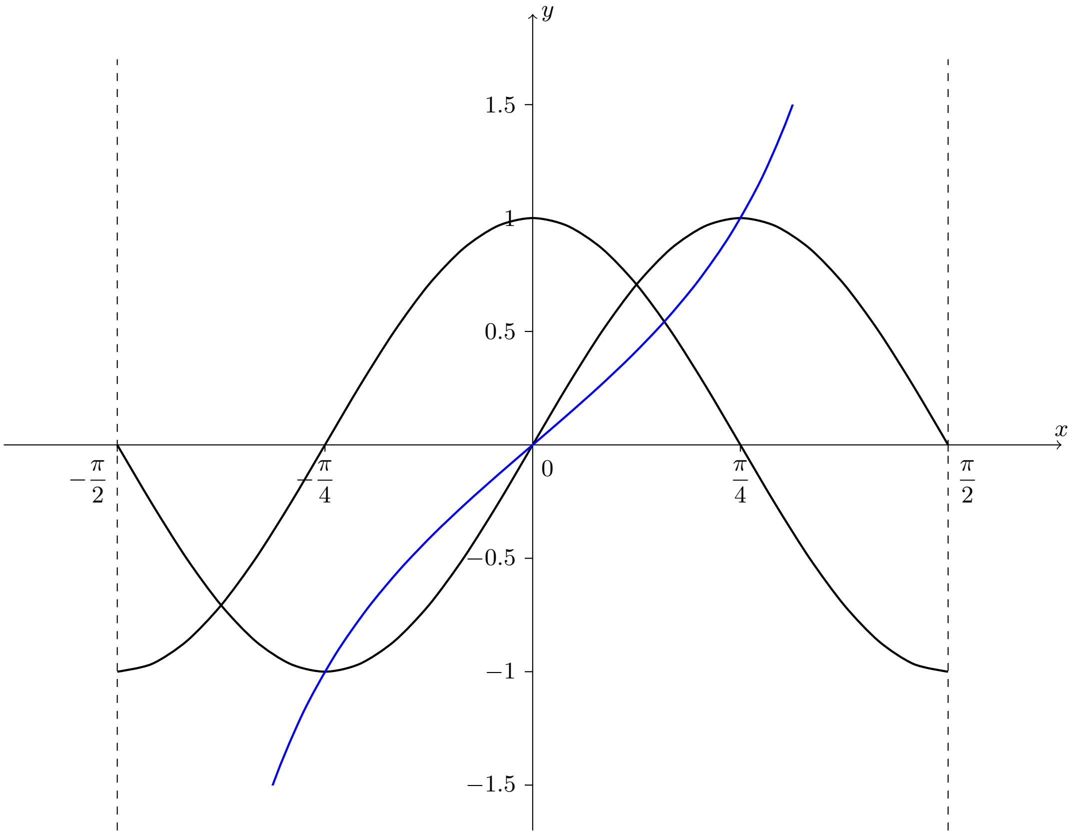
It is clear from this sketch that the area under the graph of $y = \mathrm{g}(x)$ is formed of a rectangle which is $(b-a)$ by $2\mathrm{f}(b)$ in size, and the area under the original graph, R, so that the required area is $R + (b-a) \times 2\mathrm{f}(b)$.

Question 12

We clearly need to sketch the graph of $y = \tan x$ onto the given sketch.

This graph has asymptotes at $x = \pm\frac{\pi}{2}$, and passes through the points $(\frac{\pi}{4}, 1)$ and $(-\frac{\pi}{4}, -1)$. Since the function is increasing as it passes through these points, it must lie below the graph of $y = \sin 2x$ between 0 and $\frac{\pi}{4}$, and above it between $-\frac{\pi}{4}$ and 0.

This results in the following graph:



We now pick various values for x and see which order the functions lie in. For example, when x is close to $-\frac{\pi}{2}$, we have $\tan x < \cos 2x < \sin 2x$. Here are some cases as we move from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$, each together with the option it eliminates:

x value	Function order	Option
x close to $-\frac{\pi}{2}$	$\tan x < \cos 2x < \sin 2x$	F
x just less than $-\frac{\pi}{4}$	$\tan x < \sin 2x < \cos 2x$	E
x just more than $-\frac{\pi}{4}$	$\sin 2x < \tan x < \cos 2x$	D
x just more than 0	$\tan x < \sin 2x < \cos 2x$	E
x just more than $\frac{\pi}{8}$	$\tan x < \cos 2x < \sin 2x$	F
x just less than $\frac{\pi}{4}$	$\cos 2x < \tan x < \sin 2x$	B
x just more than $\frac{\pi}{4}$	$\cos 2x < \sin 2x < \tan x$	A

The only option which is not eliminated is C: $\sin 2x < \cos 2x < \tan x$ for some real number x with $-\frac{\pi}{2} < x < \frac{\pi}{2}$

It is also clear from the graph that this is impossible: the only time that $\tan x$ is greater than both $\sin 2x$ and $\cos 2x$ is when $\frac{\pi}{4} < x < \frac{\pi}{2}$, and in the whole of that range, $\cos 2x < \sin 2x$

Question 13

One way to think about this is to write it as a column addition. If we write $a \times 10^{-3}$ as $0.00a_1a_2\ldots$, where $a = a_1.a_2a_3\ldots$ (as a decimal expansion), and similarly for the other numbers, the sum becomes:

$$\begin{array}{r} 0\,.\,0\;\;0\;\;a_1\;\;a_2\;\ldots \\ +\;0\,.\,0\;\;b_1\;\;b_2\;\;b_3\;\ldots \\ \hline 0\,.\,c_1\;\;c_2\;\;c_3\;\;c_4\;\ldots \end{array}$$

We see that, as $c_1 \geqslant 1$, there must be a carry from the hundredths column to the tenths column, and hence $b_1 = 9$, so $b \geqslant 9$. If $b = 9$ exactly, then there cannot be a carry; we would just get a sum of $0.09a_1a_2\ldots$ in that case. So we must have $b > 9$, and **II must be true**.

There is no need to have $a_1 = 9$; we could, for example, have $a_1 = 8$ and $b_2 = 3$ and still have a carry. So **I is not necessary true**.

Comparing a and b to c, we know that $1 \leqslant c < 2$, as $c_1 = 1$. We saw that $a_1 = 8$ is possible, so $a > 8$ is possible, and $a < c$ **(statement III) is not necessarily true**.

We have already shown that $b > 9$, so since $c < 2$, $b > c$ **(statement IV) is never true**.

So the answer is B.

Question 14

We can complete the square for the original quadratic:

$$x^2 - 2bx + c = (x - b)^2 - b^2 + c$$

so the vertex P is at $(b, c - b^2)$.

The graph given shows that $b > 0$ and $c - b^2 > 0$.

The new quadratic is

$$x^2 - 2Bx + c = (x - B)^2 - B^2 + c$$

so the new vertex is at $(B, c - B^2)$.

Now $B > b$, so the new vertex is to the right of P. As $B > b > 0$, $B^2 > b^2$ so $c - B^2 < c - b^2$, and the new vertex is below P.

Therefore the correct answer is F.

Question 15

From the question, it seems that we are expected to work out the first few values of the functions $\mathrm{f}(n)$ and $\mathrm{g}(n)$, and to look for a pattern.

n	$\mathrm{f}(n)$	$\mathrm{g}(n)$
1	5	3
2	16	8
3	8	4
4	4	2
5	2	1
6	1	6
7	4	3
8	2	8
9	1	4
10	4	2
11	2	1
12	1	6

So after the first three terms, $\mathrm{f}(n)$ repeats every 3 terms: for $k \geqslant 1$, $\mathrm{f}(3k+1) = 4$, $\mathrm{f}(3k+2) = 2$, $\mathrm{f}(3k+3) = 1$, while $\mathrm{g}(n)$ repeats every 6 terms, and we can write similar expressions for $\mathrm{g}(6k+r)$.

We could either calculate $\mathrm{f}(n) - \mathrm{g}(n)$ for all of these terms, and find the repeating pattern there, or find $\mathrm{f}(1000)$ and $\mathrm{g}(1000)$ separately. The latter approach seems like less effort, so we will go for that.

Note that $1000 = 3 \times 333 + 1 = 6 \times 166 + 4$. So $\mathrm{f}(1000) = 4$, $\mathrm{g}(1000) = 2$.

Thus $\mathrm{f}(1000) - \mathrm{g}(1000) = 4 - 2 = 2$ and the answer is D.

Question 16

For a function to be a counterexample to $(*)$, it must be the case that $\mathrm{f}(x)$ is an integer for every integer x, but for which $\mathrm{f}'(x)$ is **not** an integer for every integer x.

We can start by substituting in $x = 0$ and $x = 1$ into the given functions to see whether $\mathrm{f}(x)$ is even potentially an integer for every integer x:

A $\mathrm{f}(0) = \frac{1}{4}$. **This cannot be a counterexample**.

B $\mathrm{f}(0) = 0$, $\mathrm{f}(1) = \frac{3}{2}$. **This cannot be a counterexample**.

C $\mathrm{f}(0) = 0$, $\mathrm{f}(1) = 2$. **This could potentially be a counterexample**.

D $\mathrm{f}(0) = 0$, $\mathrm{f}(1) = 1$. **This could potentially be a counterexample**.

So we now only need to consider C and D.

Our next step is to differentiate them and then substitute $x = 0$ and $x = 1$ into the derivatives.

C $\mathrm{f}'(x) = \dfrac{4x^3 + 3x^2 + 2x + 1}{2}$, $\mathrm{f}'(0) = \frac{1}{2}$, so this is potentially a counterexample.

D $\mathrm{f}'(x) = \dfrac{4x^3 + 6x^2 + 2x}{4}$, $\mathrm{f}'(0) = 0$, $\mathrm{f}'(1) = 4$.

It therefore seems most likely that C is a counterexample and D is not. To prove this, we need to show that for C, $\mathrm{f}(x)$ is an integer for every integer x, and that for D, either $\mathrm{f}(x)$ is not an integer for some integer x or that $\mathrm{f}'(x)$ is an integer for every integer x.

For the purposes of the test, it probably makes most sense at this point to choose the answer C and move on, but we will go on here to complete the solution.

For C, we have

$$\mathrm{f}(x) = \frac{x(x^3 + x^2 + x + 1)}{2} = \frac{x(x+1)(x^2+1)}{2}$$

We see that $x(x+1)$ is even if x is even and also if x is odd, so $x(x+1)(x^2+1)$ is even for every integer x, hence $\mathrm{f}(x)$ is an integer for every integer x. Therefore C is a counterexample.

For D, we have

$$\mathrm{f}(x) = \frac{x^2(x^2 + 2x + 1)}{4} = \frac{x^2(x+1)^2}{4}$$

Since $x(x+1)$ is even for every integer x, as we just showed, $x^2(x+1)^2$ is a multiple of 4 for every integer x, and thus $\mathrm{f}(x)$ is an integer for every integer x.

For the derivative,

$$\mathrm{f}'(x) = \frac{x(2x^2 + 3x + 1)}{2} = \frac{x(x+1)(2x+1)}{2}$$

and as before, the numerator is always even, so $\mathrm{f}'(x)$ is an integer for every integer x. Therefore D is not a counterexample.

Question 17

We begin with the original statement with S replaced by T:

> T is stapled **if and only if for every** whole number a which is in T, **there exists** a prime factor of a which divides **at least one** other number in T.

Therefore, negating this:

> T is **not** stapled **if and only if** it is **not** the case that **for every** whole number a which is in T, **there exists** a prime factor of a which divides **at least one** other number in T.

We can negate the outer '**for every**' to get '**there exists** ... such that it is **not** the case that ...':

> T is **not** stapled **if and only if there exists** a whole number a which is in T such that it is **not** the case that **there exists** a prime factor of a which divides **at least one** other number in T.

So the answer is one of E, F, G, H.

Our current phrasing is 'it is **not** the case that **there exists**'; we could convert this into '**for every** ..., it is **not** the case that ...', but none of these options use this wording. Instead, E, F and G use the wording 'there is no prime factor', which is the same as 'there does not exist a prime factor', which in turn is the same as 'it is not the case that there exists a prime factor'. So we can rewrite the previous statement as:

> T is **not** stapled **if and only if there exists** a whole number a which is in T such that there is no prime factor of a which divides **at least one** other number in T.

Hence the correct answer is F.

Question 18

We could first check to see whether the answer makes sense. If we substitute $n = 1$ into the original equation, we get $\frac{1}{4} < \left(\frac{1}{32}\right)^{10}$, which is clearly nonsense, so there has to be a mistake somewhere in the argument.

There are two straightforward approaches to finding the mistake: one is to check that the logic of each step works, the other is to try a value of n which either does or does not solve the original inequality, and check that each step works for this value of n. Since we know that the solution is wrong, it must fail somewhere, at least when $n = 1$.

Approach 1: checking the logic

Clearly step (V) is correct, so the mistake must be earlier.

Step (I) applies the same function to both sides of an inequality, so it looks fine.

Step (II) is just a simplification of the logarithm expressions using the power rule. The powers are positive integers, and both $\frac{1}{4}$ and $\frac{1}{32}$ are positive, so this step is fine.

Starting from the other end, step (IV) is just a calculation, and we can check quite easily that it is correct.

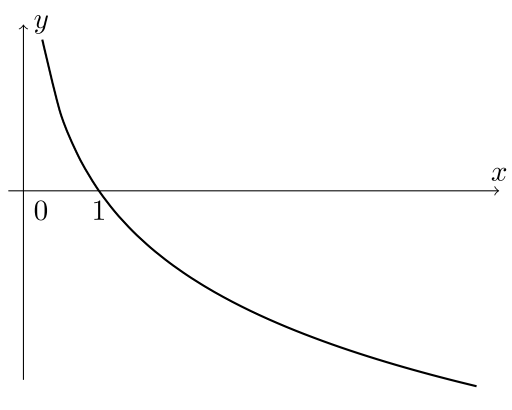
Step (III) looks like it is probably wrong, as we are performing a division with an inequality.

However, with more care, we can calculate $\log_{\frac{1}{2}}\left(\frac{1}{4}\right) = 2$ (as we need to do to check step (IV) anyway), so step (III) is just dividing by 2, and is actually fine.

Unfortunately, though, every step looks fine.

Working through them once more, steps (IV) and (V) are certainly fine, as is step (III). We have argued that step (II) is correct, so by elimination it must be step (I) which is wrong.

It is helpful to sketch a graph of $y = \log_{\frac{1}{2}} x$ to understand why step (I) is wrong. We have $\log_{\frac{1}{2}}(\frac{1}{2}) = 1$, $\log_{\frac{1}{2}} 1 = 0$, $\log_{\frac{1}{2}} 2 = -1$, so the graph looks like this:



So $\log_{\frac{1}{2}} x$ is a *decreasing* function: if $x < y$, then $\log_{\frac{1}{2}} x > \log_{\frac{1}{2}} y$; in other words, $\log_{\frac{1}{2}}$ reverses the direction of an inequality. Therefore step (I) is incorrect: the inequality should have been reversed in this step.

The correct answer is therefore B.

Approach 2: substituting a value of n

We have already shown that $n = 1$ does not satisfy the original inequality, yet it is given as a solution at the end of step (V). So we use this value of n to find at least one error in the argument.

Note that this approach is not guaranteed to work: there may be other errors in the argument that will escape detection by this method, but given that the options tell us that only one step is invalid, it will be sufficient for our purposes.

It will help us with our calculations if we write everything in terms of powers of $\frac{1}{2}$.

The original inequality can be written as

$$\left(\frac{1}{2}\right)^{2n} < \left(\frac{1}{2}\right)^{50}$$

Substituting $n = 1$ gives

$$\left(\frac{1}{2}\right)^{2} < \left(\frac{1}{2}\right)^{50}$$

which is *not* true.

After step (I), the inequality reads

$$\log_{\frac{1}{2}} \left(\frac{1}{2}\right)^{2n} < \log_{\frac{1}{2}} \left(\frac{1}{2}\right)^{50}$$

Substituting $n = 1$ gives

$$\log_{\frac{1}{2}} \left(\frac{1}{2}\right)^{2} < \log_{\frac{1}{2}} \left(\frac{1}{2}\right)^{50}$$

which we can simplify to

$$2 < 50$$

which *is* true.

So something has gone wrong in step (I), and this is where the error lies.

> ***Commentary:*** *There is a slightly subtle question about whether the student intended each of the steps in this argument to be reversible. For example, if a student started with the statement $x + 5 = 8$ and deduced $x > 0$, they have made a correct deduction (though not 'solved the equation'), but the argument is not reversible: $x > 0$ does not imply that $x + 5 = 8$. So we will assume that in this case they intend to say that $1 \leqslant n \leqslant 24$ are all solutions to the original equation.*
>
> *It follows that step (I) is where the mistake lies, as $n = 1$ satisfies the inequality after step (I) but not the inequality before it.*
>
> *If we did not make this assumption about the student's intentions, we could repeat the argument above with a value of n which* does *satisfy the original inequality, such as $n = 100$; then the first inequality is true but the second inequality, obtained after step (I), is false, so the proposed implication is false.*

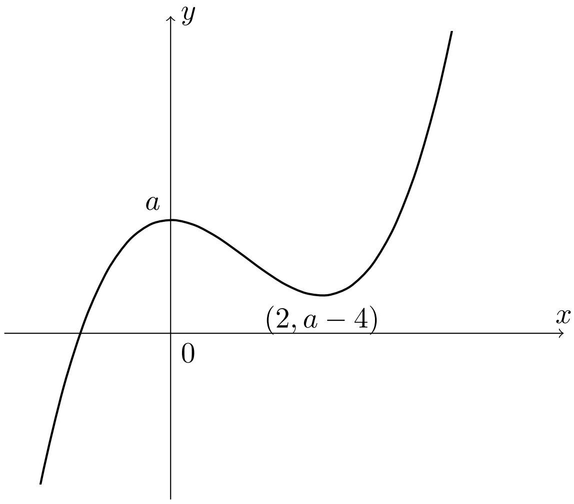
Question 19

We recall that a '**sufficient** condition' means that if the condition holds, then the equation has exactly one real root.

We can start by sketching the graph of the cubic $y = x^3 - 3x^2 + a$, noting that the impact of changing a is just to translate the graph vertically.

We differentiate to find the stationary points: $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x$, so $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$ and $x = 2$, so the stationary points are at $(0, a)$ and $(2, a - 4)$.

So the graph looks like this:



There will be one real root for the equation $x^3 - 3x^2 + a = 0$ if either the local maximum lies below the x-axis or if the local minimum lies above the x-axis, as in the case sketched.

The first occurs when $a < 0$, the second when $a - 4 > 0$ or equivalently, when $a > 4$.

So a **necessary and sufficient** condition is $a < 0$ or $a > 4$. This does not, however, appear as one of the options, so we need to go through them to see which one is **sufficient**. (Recall from earlier that **sufficient** means that if this condition holds, then the equation has exactly one real root, but not necessarily the other way round.)

A This includes $a = 1$, which gives more than one real root.

B This includes $a = 0$, which gives more than one real root (as the minimum lies on the x-axis).

C This includes $a = 4$, which gives more than one real root (as the maximum lies on the x-axis).

D This includes $a = 1$ again.

E This is $a > 4$ or $a < -4$, both of which are part of the valid range, so this is a **sufficient** condition.

F This includes $a = 1$ again.

G This gives more than one real root.

H $a = \frac{3}{2}$ gives more than one real root.

So the correct answer is E.

Question 20

We can write this out in a grid to see what the current state of knowledge is.

	Position 1	2	3	4	5
Tried	a	b	c	d	e
	c	d	b	e	a
	e	a	d	b	c
Possible	b	c	a	a	b
	d	e	e	c	d

So positions 1 and 5 contain `b` and `d` in some order, while positions 2, 3, 4 contain either `cea` or `eac` in that order.

We certainly do not know the correct password yet.

We could sensibly try `bcead` as our next attempt. What might we be told in response?

If we have the `b` and `d` in the correct order, these 2 letters will be correct, otherwise neither is correct.

If we have the middle three letters `cea` in the correct order, then these 3 letters will be correct, otherwise none of them is.

So:

- if both are correct, the computer will tell us that 5 letters are correct, and we are done;
- if the outer pair are correct but the middle ones are not, the computer will tell us that 2 letters are correct, and we can deduce that the correct password is `beacd`;
- if the outer pair are wrong but the middle ones are correct, the computer will tell us that 3 letters are correct, and we can deduce that the correct password is `dceab`;
- if both are wrong, the computer will tell us that 0 letters are correct, and we can deduce that the correct password is `deacb`.

Since the four cases give four distinct responses by the computer (0, 2, 3, 5 correct letters), we can distinguish between them, and therefore deduce the correct password with just this one further attempt.

The correct answer is B.

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