Cambridge Assessment
Admissions Testing

TEST OF MATHEMATICS FOR UNIVERSITY ADMISSION **D513/01**

PAPER 1

Wednesday 30 October 2019 Written solutions **75 minutes**

Additional materials: Answer sheet

INSTRUCTIONS TO CANDIDATES

Please read these instructions carefully, but do not open the question paper until you are told that you may do so.

A separate answer sheet is provided for this paper. Please check you have one. You also require a soft pencil and an eraser.

This paper is the first of two papers.

There are 20 questions on this paper. For each question, choose the one answer you consider correct and record your choice on the separate answer sheet. If you make a mistake, erase thoroughly and try again.

There are no penalties for incorrect responses, only marks for correct answers, so you should attempt **all** 20 questions. Each question is worth one mark.

You can use the question paper for rough working or notes, but **no extra paper** is allowed.

Please complete the answer sheet with your candidate number, centre number, date of birth, and full name.

You **must** complete the answer sheet within the time limit.

Calculators and dictionaries are NOT permitted.

There is no formulae booklet for this test.

Please wait to be told you may begin before turning this page.

This question paper consists of 21 printed pages and 3 blank pages.

PV2

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1 f(x) is a quadratic function in x.

The graph of $y = \mathrm{f}(x)$ passes through the point $(1, -1)$ and has a turning point at $(-1, 3)$.

Find an expression for f(x).

A $-x^2 - 2x + 2$

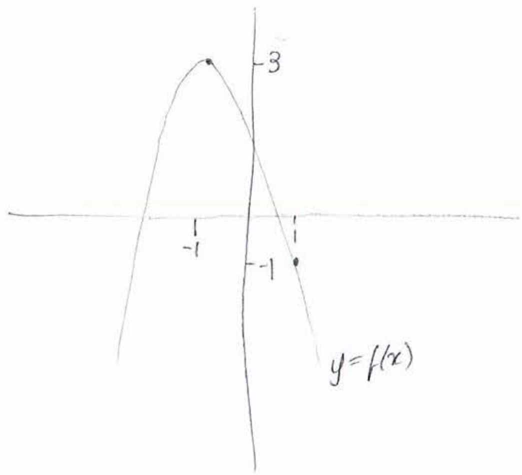
B $-x^2 + 2x + 3$

C $x^2 - 2x$

D $x^2 + 2x - 4$

E $2x^2 + 4x + 1$

F $-2x^2 - 4x + 5$



let $y = Ax^2 + Bx + C$

Turning point where $\frac{dy}{dx} = 2Ax + B = 0$ so $2Ax = -B$

$x = \frac{-B}{2A} = -1$

so $B = 2A$ and $y = Ax^2 + 2Ax + C$

Passes through $(-1, 3)$ and $(1, -1)$ so $3 = A - 2A + C$

$3 = C - A \Rightarrow C = A + 3$

and $-1 = A + 2A + C$

$-1 = 3A + C \Rightarrow C = -1 - 3A$

$A + 3 = -1 - 3A$

$4A = -4$

$A = -1$ so $C = -1 + 3 = 2$ and $B = 2 \times -1 = -2$

so $y = -x^2 - 2x + 2$

2 Find the complete set of values of the real constant k for which the expression

$$x^2 + kx + 2x + 1 - 2k = x^2 + (2+k)x + (1-2k)$$

→ no real roots

is positive for all real values of x.

A $-12 < k < 0$

B $k < -12$ or $k > 0$

C $-\sqrt{6} - 3 < k < \sqrt{6} - 3$

D $k < -\sqrt{6} - 3$ or $k > \sqrt{6} - 3$

E $-2 < k < \frac{1}{2}$

F $k < -2$ or $k > \frac{1}{2}$

G $0 < k < 4$

H $k < 0$ or $k > 4$

Discriminant $= (2+k)^2 - 4(1-2k) < 0$

$k^2 + 4k + 4 - 4 + 8k < 0$

$k^2 + 12k < 0$

$k(k+12) < 0$

so $\underline{-12 < k < 0}$

3 Find the coefficient of x in the expression:

$$(1+x)^0 + (1+x)^1 + (1+x)^2 + (1+x)^3 + \cdots + (1+x)^{79} + (1+x)^{80}$$

$0 \quad 1 \quad 2 \quad 3 \quad 79 \quad 80$

A 80

B 81

C 324

D 628

E 3240

F 3321

G 6480

H 6642

x coefficient in $(1+x)^n$ is n

so coefficient is $0+1+2+3+\ldots+79+80$

Arithmetic progression first term 0, common difference 1

Sum of AP $= \frac{1}{2}n(n+1) = \frac{1}{2} \times 80 \times 81 = 40 \times 81 = \underline{\underline{3240}}$

4 The sequence x_n is given by:

$$x_1 = 10$$

$$x_{n+1} = \sqrt{x_n} \quad \text{for } n \geq 1$$

$= (x_n)^{1/2}$

What is the value of x_{100}?

[Note that a^{b^c} means $a^{(b^c)}$]

so $x_2 = x_1^{1/2} = 10^{1/2}$

$x_3 = x_2^{1/2} = (10^{1/2})^{1/2} = 10^{(1/2)^2}$

$x_4 = x_3^{1/2} = ((10^{1/2})^{1/2})^{1/2} = 10^{(1/2)^3}$

$\therefore x_{100} = 10^{(1/2)^{99}} = 10^{1/2^{99}} = \underline{\underline{10^{2^{-99}}}}$

A $10^{2^{99}}$

B $10^{2^{100}}$

C $10^{2^{-99}}$

D $10^{2^{-100}}$

E $10^{-2^{99}}$

F $10^{-2^{100}}$

G $10^{-2^{-99}}$

H $10^{-2^{-100}}$

5 S is a geometric sequence.

The sum of the first 6 terms of S is equal to 9 times the sum of the first 3 terms of S.

The 7$^{\text{th}}$ term of S is 360.

Find the 1$^{\text{st}}$ term of S. Geometric progression so $a, ar, ar^2, \ldots$

A $\frac{40}{27}$

B $\frac{40}{9}$

C $\frac{40}{3}$

D $\frac{45}{16}$

E $\frac{45}{8}$

F $\frac{45}{4}$

Sum of first 6 terms $= a + ar + ar^2 + ar^3 + ar^4 + ar^5$
$= a(1+r+r^2+r^3+r^4+r^5)$

Sum of first 3 terms $= a + ar + ar^2$
$= a(1+r+r^2)$

7th term $\boxed{360 = ar^6}$

$$S_n = \frac{a(r^n-1)}{r-1}$$

$$S_6 = \frac{a(r^6-1)}{r-1} \quad \text{and} \quad 9S_3 = \frac{9a(r^3-1)}{r-1}$$

$$\frac{\cancel{a}(r^6-1)}{\cancel{r-1}} = \frac{9\cancel{a}(r^3-1)}{\cancel{r-1}}$$

$$r^6 - 1 = 9r^3 - 9$$

$$\boxed{r^6 + 8 = 9r^3}$$

let $y = r^3$ then $y^2 - 9y + 8 = 0$
$(y-8)(y-1) = 0$ so $r^3 = 8$ ~~or 1~~

so $r = 2$ and $360 = 2^6 \times a = 64a$ so $a = \frac{360}{64} = \frac{180}{32} = \frac{90}{16} = \frac{45}{8}$

6 The circles with equations

$$(x+4)^2 + (y+1)^2 = 64 \quad \text{and}$$ centre $(-4,1)$ radius 8

$$(x-8)^2 + (y-4)^2 = r^2 \quad \text{where } r > 0$$ centre $(8,4)$ radius $r>0$

have exactly one point in common.

Find the difference between the two possible values of r.

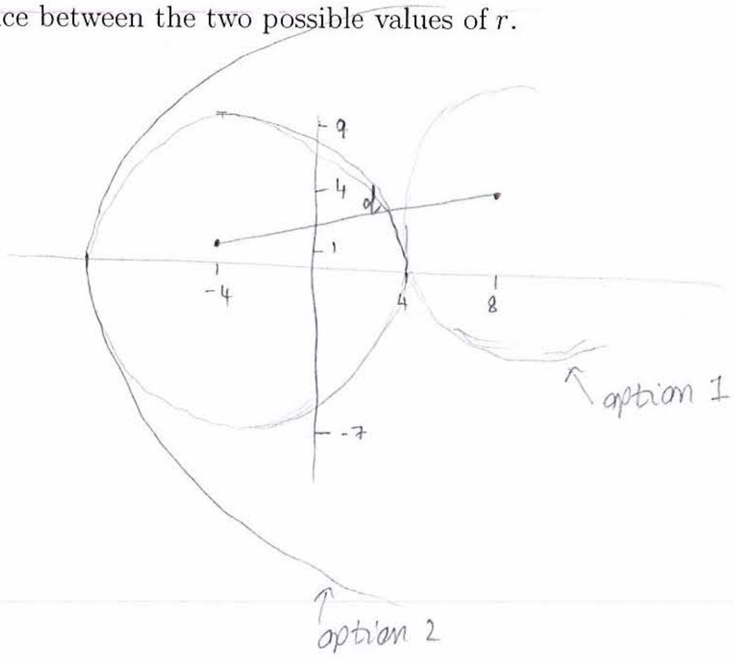
A 4

B 10

C 16

D 26

E 50

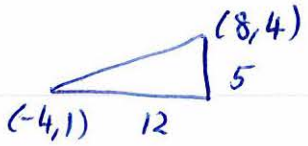


One point in common if they touch but don't intersect

d = distance between 2 centres

so $d = 8 + r$ or $d = r - 8$

$r = d - 8$ $\qquad$ $r = d + 8$



so $d = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = 13$

so $r = 13 \pm 8 = 5$ or 21

$21 - 5 = \underline{16}$

7 A curve has equation

$$y = (2q - x^2)(2qx + 3) = 4q^2x + 6q - 2qx^3 - 3x^2$$

The gradient of the curve at $x = -1$ is a function of q. $= -2qx^3 + (4q^2 - 3)x^2 + 6q$

Find the value of q which minimises the gradient of the curve at $x = -1$.

A -1 $\frac{dy}{dx} = 4q^2 - 6qx^2 - 6x$

B $-\frac{3}{4}$

C $-\frac{1}{2}$ Gradient at $x = -1$ is $4q^2 - 6q - 6$

D 0

E $\frac{1}{2}$ $\frac{d}{dq}\left(\frac{dy}{dx}\right) = 8q - 6$ at $x = -1$

F $\frac{3}{4}$

G 1 Max or min when $8q - 6 = 0$

$8q = 6$

$q = 3/4$

8 The function f is such that $0 < \mathrm{f}(x) < 1$ for $0 \leq x \leq 1$.

The trapezium rule with n equal intervals is used to estimate $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x$ and produces an underestimate.

Using the same number of equal intervals, for which one of the following does the trapezium rule produce an overestimate?

A $\int_0^1 (\mathrm{f}(x) + 1)\,\mathrm{d}x$

B $\int_0^1 2\mathrm{f}(x)\,\mathrm{d}x$

C $\int_{-1}^0 \mathrm{f}(x + 1)\,\mathrm{d}x$

D $\int_{-1}^0 \mathrm{f}(-x)\,\mathrm{d}x$

E $\int_0^1 (1 - \mathrm{f}(x))\,\mathrm{d}x$

same underestimation as original expression

underest. 2x as much as original

$\int_0^1 (1-f(x))\,dx$ and $\int_0^1 f(x)\,dx$ added together give the unit square, so if one is an underestimate the other must be an overestimate

9 p is a positive constant.

Find the area enclosed between the curves $y = p\sqrt{x}$ and $x = p\sqrt{y}$

A $\frac{2}{3}p^{\frac{5}{2}} - \frac{1}{2}p^2$

B $\frac{4}{3}p^{\frac{5}{2}} - p^2$

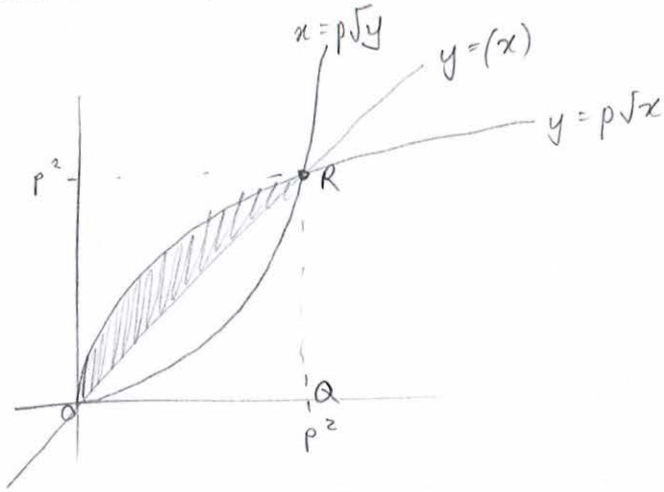
C $\frac{p^4}{6}$

D $\frac{p^4}{3}$

E $\frac{2}{3}p^3 - \frac{1}{2}p^4$

F $\frac{4}{3}p^3 - p^4$

G $2p^4$



curves intersect at $(0, 0)$ and (p^2, p^2)
symmetrical about $y = x$

Area under $y = p\sqrt{x}$ above x axis

$$\int_0^{p^2} p\sqrt{x}\,dx = p\int_0^{p^2} x^{1/2}\,dx = p\left(\frac{x^{3/2}}{3/2}\Big|_0^{p^2}\right) = p\left(\frac{2}{3}(p^2)^{3/2}\right)$$

$$= p \times \frac{2}{3} \times p^3 = \frac{2}{3}p^4$$

Area of shaded region $= \frac{2}{3}p^4$ − area of Δ OQR

$$= \frac{2}{3}p^4 - \left(\frac{1}{2} \times p^2 \times p^2\right)$$

$$= \frac{2}{3}p^4 - \frac{1}{2}p^4 = \frac{1}{6}p^4$$

So area enclosed between curves $= 2 \times \frac{1}{6}p^4 = \frac{1}{3}p^4 = \underline{\underline{\frac{p^4}{3}}}$

10 Evaluate

$$\int_{-1}^{3} |x|(1-x)\,\mathrm{d}x$$

A $\frac{17}{3}$

B $-\frac{17}{3}$

C $\frac{16}{3}$

D $-\frac{16}{3}$

E $\frac{11}{3}$

F $-\frac{11}{3}$

When $-1 \leq x < 0$, $|x|(1-x) = -x(1-x)$

$= -x + x^2$

When $0 \leq x < 3$, $|x|(1-x) = x(1-x)$

$= x - x^2$

so $\int_{-1}^{3} |x|(1-x)\,dx = \int_{1}^{0} -x^2 + x^2\,dx + \int_{0}^{3} x - x^2\,dx$

$= -\frac{x^2}{2} + \frac{x^3}{3}\Big|_{-1}^{0} + \frac{x^2}{2} - \frac{x^3}{3}\Big|_{0}^{3}$

$= \left(0 - \left(-\frac{1}{2} - \frac{1}{3}\right)\right) + \left(\frac{9}{2} - \frac{27}{3} - 0\right)$

$= \frac{5}{6} - \frac{9}{2} = \frac{-11}{3}$

11 Find the sum of the real values of x that satisfy the simultaneous equations:

$$\log_3(xy^2) = 1$$

$$\log_3(x) + 2\log_3(y) = 1$$

$$(\log_3 x)(\log_3 y) = -3$$

A $\frac{1}{3}$

B 1

C 3

D $3\frac{1}{9}$

E $9\frac{1}{27}$

F $9\frac{1}{3}$

G 27

H $27\frac{1}{9}$

let $u = \log_3 x$ then $u + 2v = 1$ and $uv = -3$

$v = \log_3 y$ $u = 1 - 2v$ $u = \frac{-3}{v}$

so $1 - 2v = \frac{-3}{v}$

$v - 2v^2 = -3$

$2v^2 - v - 3 = 0$ $v = \frac{1 \pm \sqrt{1 + 4 \times 2 \times 3}}{4} = \frac{1 \pm 5}{4} = 3/2$ or -1

so $u = -2$ or 3

$\therefore \log_3 x = -2$ and $x = 1/9$ or $\log_3 x = 3$ and $x = 27$

$27 + 1/9 = \underline{\underline{27\,1/9}}$

12 It is given that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{24\pi(t-1)}{(1+\sqrt{t}\,)} \text{ for } t \geq 1$$

and $V = 7$ when $t = 1$.

Find the value of V when $t = 9$.

$$= \frac{24\pi(t-1)(1-\sqrt{t})}{(1+\sqrt{t})(1-\sqrt{t})}$$

$$= \frac{24\pi(t-1)(1-\sqrt{t})}{1-t}$$

$$= \frac{-24\pi\cancel{(1-t)}(1-\sqrt{t})}{\cancel{1-t}}$$

$$= 24\pi(\sqrt{t}-1)$$

$$= 24\pi(t^{1/2}-1)$$

A $208\pi + 7$

B $216\pi + 7$

C $224\pi + 7$

D $416\pi + 7$

E $608\pi + 7$

F $744\pi + 7$

$$V = \int 24\pi(t^{1/2}-1)\,dt = 24\pi\left(\frac{2t^{3/2}}{3} - t\right) + c$$

When $t = 1$, $V = 7$ so:

$$7 = 24\pi\left(2/3 - 1\right) + c$$

$$7 = -\frac{24}{3}\pi + c$$

$$7 = -8\pi + c$$

$$c = 7 + 8\pi$$

$\therefore$ $V = 24\pi\left(\frac{2t^{3/2}}{3} - t\right) + 7 + 8\pi$

When $t = 9$, $V = 24\pi\left(\frac{2 \times 27}{3} - 9\right) + 7 + 8\pi$

$= 24\pi \times 9 + 7 + 8\pi$

$= 216\pi + 8\pi + 7$

$= \underline{\underline{224\pi + 7}}$

13 Find the maximum value of

$$f(x) = 4^{\sin x} - 4 \times 2^{\sin x} + \tfrac{17}{4}$$

$$= (2^2)^{\sin x} - 2^2 \times 2^{\sin x} + 17/4$$

$$= (2^{\sin x})^2 - 4 \times 2^{\sin x} + 17/4$$

$$= (2^{\sin x} - 2)^2 - 4 + 17/4$$

$$= (2^{\sin x} - 2)^2 + 1/4$$

for real x.

A $\frac{1}{4}$

B $\frac{5}{2}$

C $\frac{13}{2}$

D $\frac{21}{2}$

E $\frac{65}{4}$

F There is no maximum value.

$-1 \leq \sin x \leq 1$ so $1/2 \leq 2^{\sin x} \leq 2$

so $-3/2 \leq (2^{\sin x} - 2) \leq 0$ & the max of $(2^{\sin x} - 2)^2$ is $9/4$

so max of $f(x)$ is $9/4 + 1/4 = \underline{\underline{5/2}}$

14 x satisfies the simultaneous equations

$$\sin 2x + \sqrt{3}\cos 2x = -1$$

$\sin 2x = -1 - \sqrt{3}\cos 2x$

and
$$\sqrt{3}\sin 2x - \cos 2x = \sqrt{3}$$

$\sqrt{3}\sin 2x = \sqrt{3} + \cos 2x$

where $0° \leq x \leq 360°$.

$\sin 2x = \frac{\sqrt{3} + \cos 2x}{\sqrt{3}} = 1 + \frac{\sqrt{3}\cos 2x}{3}$

Find the sum of the possible values of x.

A 210°

B 330°

C 390°

D 660°

E 780°

F 930°

$-1 - \sqrt{3}\cos 2x = 1 + \frac{\sqrt{3}\cos 2x}{3}$

$-3 - 3\sqrt{3}\cos 2x = 3 + \sqrt{3}\cos 2x$

$-6 = 4\sqrt{3}\cos 2x$

$\cos 2x = \frac{-6}{4\sqrt{3}} = \frac{-3}{2\sqrt{3}} = \frac{-3\sqrt{3}}{6} = -\frac{\sqrt{3}}{2}$

so $\sin 2x = -1 - \sqrt{3} \times (-\sqrt{3}/2) = -1 + 3/2 = 1/2$

$\cos 2x = -\frac{\sqrt{3}}{2} \Rightarrow 2x = 150°, 210°, 510°, 570°$

$x = 75°, 105°, 255°, 285°$

$\sin 2x = \frac{1}{2} \Rightarrow 2x = 30°, 150°, 390°, 510°$

$x = 15°, 75°, 195°, 255°$

As x needs to satisfy both equations, we need $x = 75°, 255°$

$75° + 255° = \underline{\underline{330°}}$

15 Find the real non-zero solution to the equation

$$\frac{2^{(9^x)}}{8^{(3^x)}} = \frac{1}{4}$$

A $\log_3 2$

B $2\log_3 2$

C 1

D 2

E $\log_2 3$

F $2\log_2 3$

$$\frac{2^{(9^x)}}{2^{3(3^x)}} = \frac{1}{4}$$

$$2^{(9^x - 3(3^x))} = 2^{-2}$$

$$9^x - 3(3^x) = -2$$

$$(3^2)^x - 3(3^x) = -2$$

$$(3^x)^2 - 3(3^x) + 2 = 0$$

$$(3^x - 1)(3^x - 2) = 0$$

Need $3^x = 1$ or $3^x = 2$

$\cancel{x=0}$

$x = \underline{\underline{\log_3 2}}$

16 Given that

$$2\int_0^1 f(x)\,dx + 5\int_1^2 f(x)\,dx = 14$$

and

$$\int_0^1 f(x+1)\,dx = 6$$

$= \int_1^2 f(x)\,dx$

find the value of

$$\int_0^2 f(x)\,dx$$

A -8

B -4

C -2

D 2

E 4

F $\frac{29}{5}$

G $\frac{32}{5}$

H 14

$2\int_0^1 f(x)\,dx + 5\int_1^2 f(x)\,dx = 14$

$\underbrace{2\int_0^1 f(x)\,dx + 2\int_1^2 f(x)\,dx}_{2\int_0^2 f(x)\,dx} + 3\int_1^2 f(x)\,dx = 14$

$2\int_0^2 f(x)\,dx + 3 \times 6 = 14$

$2\int_0^2 f(x)\,dx = -4$

$\int_0^2 f(x)\,dx = \underline{\underline{-2}}$

17 Find the fraction of the interval $0 \le \theta \le \pi$ for which the inequality

$$\left(\sin(2\theta) - \tfrac{1}{2}\right)\left(\sin\theta - \cos\theta\right) \ge 0$$

is satisfied.

A $\frac{1}{12}$

B $\frac{1}{6}$

C $\frac{1}{4}$

D $\frac{5}{12}$

E $\frac{7}{12}$

F $\frac{3}{4}$

G $\frac{5}{6}$

H $\frac{11}{12}$

where both factors +ve

$\sin\theta - \cos\theta \geq 0 \Rightarrow \pi/4 \le \theta \le \pi$

$\sin 2\theta - 1/2 \geq 0 \Rightarrow \pi/12 \le \theta \le 5\pi/12$

These overlap for $\pi/4 \le \theta \le \frac{5\pi}{12}$

$\frac{5\pi}{12} - \frac{\pi}{4} = \frac{\pi}{6}$

where both factors -ve

$\sin\theta - \cos\theta \le 0 \Rightarrow 0 \le \theta \le \pi/4$

$\sin 2\theta - 1/2 \le 0 \Rightarrow 0 \le \theta \le \pi/12$ and $\frac{5\pi}{12} \le \theta \le \pi$

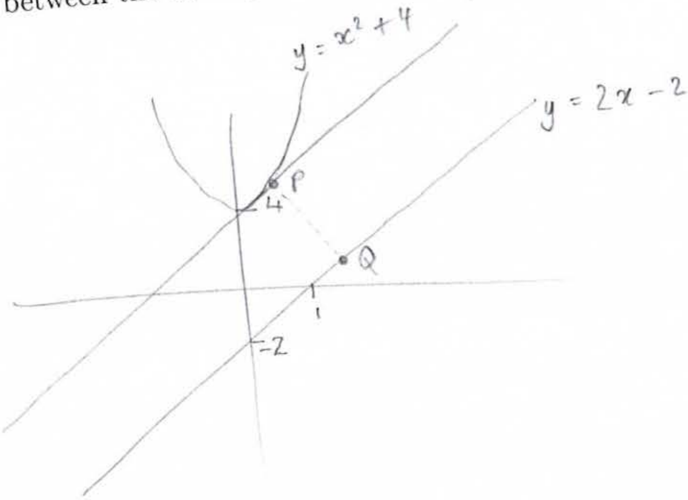
These overlap for $0 \le \theta \le \pi/12$

$\frac{\pi}{12} - 0 = \frac{\pi}{12}$

Total length where inequality satisfied $= \frac{\pi}{6} + \frac{\pi}{12} = \frac{\pi}{4}$

This is $\underline{1/4}$ of the interval $0 \le \theta \le \pi$

18 Find the shortest distance between the curve $y = x^2 + 4$ and the line $y = 2x - 2$.



A 2

B $\sqrt{5}$

C $\frac{6\sqrt{5}}{5}$

D 3

E $\frac{5\sqrt{5}}{3}$

F 5

G 6

Need tangent to curve parallel to line

Gradient of line = 2

$\frac{dy}{dx} = 2x$ so need $2x = 2$, $x = 1$

so when $x = 1$, $y = x^2 + 4 = 1 + 4 = 5$ so relevant point is $P(1,5)$

Normal that passes through P has gradient $-1/2$ & point $(1,5)$

ie $y = mx + c$ so $5 = -1/2 + c$, $c = 5\frac{1}{2}$ so $y = -\frac{x}{2} + \frac{11}{2}$ and $2x = y + 2$, $x = \frac{y}{2} + 1$

$2y = 11 - x$

$x = 11 - 2y$

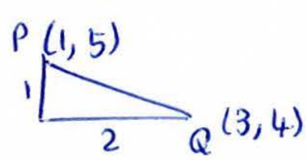
so $\frac{y}{2} + 1 = 11 - 2y$

$y + 2 = 22 - 4y$

$5y = 20$

$y = 4$

and $x = 11 - 8 = 3$ so Q is $(3,4)$



so shortest distance $= \sqrt{2^2 + 1^2} = \underline{\underline{\sqrt{5}}}$

19 Find the value of

$$\sum_{k=0}^{90} \sin(10+90k)^\circ$$

A 0

B $\sin 10^\circ$

C $\sin 100^\circ$

D $\sin 190^\circ$

E $\sin 280^\circ$

F 1

$= \sin 10^\circ + \sin(10+90)^\circ + \sin(10+180)^\circ + \sin(10+270)^\circ + \dots$

since $\sin 10^\circ = -\sin(10+180)^\circ$

$\sin(10+90)^\circ = -\sin(10+270)^\circ$

so $u_0 + u_1 + u_2 + u_3 = 0$

And $u_k = u_{k+4}$ so $\sum_{k=0}^{90} \sin(10+90k)^\circ = u_{88} + u_{89} + u_{90}$

$= \sin 10^\circ + \sin(10+90)^\circ + \sin(10+180)^\circ$

$= \underline{\underline{\sin 100^\circ}}$

20 What is the complete range of values of k for which the curves with equations

$$y = x^3 - 12x$$

and

$$y = k - (x-2)^2$$

intersect at **three** distinct points, of which exactly **two** have positive x-coordinates?

A $-4 < k < 0$

B $-4 < k < 4$

C $-4 < k < 16$

D $-16 < k < 0$

E $-16 < k < 4$

F $-16 < k < 16$

$y = x^3 - 12x$ is roughly

Rotational symmetry 0

$\frac{dy}{dx} = 3x^2 - 12$

when $\frac{dy}{dx} = 0$, $12 = 3x^2$

$x = \pm 2$ local minimum

Local min. must be +ve so local min. at $x = 2$ and $y = 2^3 - 12 \times 2$
$= -16$

$(2, -16)$

$y = k - (x-2)^2$ is roughly

symmetry about $x = 2$

Different values of k translate parabola vertically to have the vertex at $(2, k)$

Parabola less steep than cubic at the 'edges' so 3 distinct roots (2 +ve) for k between those when parabola touches local min. of cubic ($k = -16$) and when parabola passes through origin ($k = 4$).

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