GCE A LEVEL MARKING SCHEME

SUMMER 2022

A LEVEL (NEW)
FURTHER MATHEMATICS
UNIT 4 FURTHER PURE MATHEMATICS B
1305U40-1

INTRODUCTION

This marking scheme was used by WJEC for the 2022 examination. It was finalised after detailed discussion at examiners' conferences by all the examiners involved in the assessment. The conference was held shortly after the paper was taken so that reference could be made to the full range of candidates' responses, with photocopied scripts forming the basis of discussion. The aim of the conference was to ensure that the marking scheme was interpreted and applied in the same way by all examiners.

It is hoped that this information will be of assistance to centres but it is recognised at the same time that, without the benefit of participation in the examiners' conference, teachers may have different views on certain matters of detail or interpretation.

WJEC regrets that it cannot enter into any discussion or correspondence about this marking scheme.

WJEC GCE A LEVEL FURTHER MATHEMATICS

UNIT 4 FURTHER PURE MATHEMATICS B

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1. a)	Differentiating $f'(x) = 3\cosh^2 x \sinh x - 3\sinh x$ oe At a stationary point, $f'(x) = 0$, $\therefore\ 3\cosh^2 x \sinh x - 3\sinh x = 0$ $3\sinh x\,(\cosh^2 x - 1) = 0$ THEN $3\sinh x = 0$ $x = 0$ or $\cosh^2 x - 1 = 0$ $\cosh x = 1$ or $\cosh x = -1$ $x = 0$ no solutions $\therefore$ The only stationary point is at $x = 0$. OR As $\cosh^2 x - 1 = \sinh^2 x$ $3\sinh x\,(\cosh^2 x - 1) = 3\sinh^3 x = 0$ $\therefore \sinh^3 x = 0$ $\sinh x = 0$ $x = 0$ $\therefore$ The only stationary point is at $x = 0$.	M1 A1 m1 A1 A1 (A1) (A1)	If identities used, must be a valid attempt at differentiation Award for any solution of hyperbolic equation Must be seen to discard equations with no solutions and show all remaining equations lead to $x = 0$ Correct use of identity

b)	METHOD 1 Find gradient or value of f either side of $x = 0$ e.g. $f'(-1) = -4.869 \ldots < 0$ and $f'(1) = 4.869 \ldots > 0$ e.g. $\sinh x < 0$ and therefore $\sinh^3 x < 0$ for $x < 0$ and $\sinh x > 0$ and therefore $\sinh^3 x > 0$ for $x > 0$ Therefore, the stationary point at $x = 0$ is a minimum METHOD 2 Differentiating and substituting $x = 0$ $f''(x) = 3\cosh^3 x + 6\cosh x \sinh^2 x - 3\cosh x$ oe $f''(0) = 3 + 0 - 3 = 0$ Finding the gradient either side of $x = 0$ AND Stating the stationary point at $x = 0$ is a minimum	M1 A1 (M1) (A1)	Accept graphical method cao cao
c)	When $x = 0$, $f(x) = -2$ Therefore, largest range is $[-2, \infty)$	 B1	 Allow ‘Range $f(x) \geq -2$’

2.	Let $z^4 = 9 - 3\sqrt{3}\mathrm{i}$		
	$\lvert z^4\rvert = \sqrt{9^2 + \left(3\sqrt{3}\right)^2} = \sqrt{108}$ or $6\sqrt{3}$	B1	si
	Finding the radius of the circle	M1	FT their $\lvert z^4\rvert$
	e.g. Radius of circle = $\sqrt[8]{108}$ or $108^{\frac{1}{8}}$ $= 1.795\ldots$	A1	
	Circle: $x^2 + y^2 = 3.22$ or 1.795^2	A1	FT their radius Allow 1.8^2 Allow $\lvert z\rvert = 108^{1/8}$
3. a)	Substituting $\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$ $4 \times \frac{2t}{1+t^2} + 5 \times \frac{1-t^2}{1+t^2} = 3$	M1	
	$4 \times 2t + 5(1-t^2) = 3(1+t^2)$ oe $8t + 5 - 5t^2 = 3 + 3t^2$	A1	Removal of fractions
	$4t^2 - 4t - 1 = 0$	A1	convincing
b)	Solving $4t^2 - 4t - 1 = 0$ $t = \frac{1\pm\sqrt{2}}{2}$ (–0.207106... or 1.207106...)	M1 A1	M0A0 no working
	Attempting to solve for θ $\tan\frac{\theta}{2} = \frac{1-\sqrt{2}}{2}$ or $\tan\frac{\theta}{2} = \frac{1+\sqrt{2}}{2}$	M1	FT their t
	$\frac{\theta}{2} = -11.7\ldots(+180n)$ or	A1	$\frac{\theta}{2} = -0.2\ldots(+\pi n)$
	$\frac{\theta}{2} = 50.36\ldots(+180n)$	(A1)	$\frac{\theta}{2} = 0.87\ldots(+\pi n)$
	Then, the general solution, $\theta = (-23.4(018\ldots) + 360n)°$ oe or	A1	$\theta = (-0.408\ldots + 2\pi n)^c$
	$\theta = (100.7(214\ldots) + 360n)°$ oe	A1	$\theta = (1.758\ldots + 2\pi n)^c$ M0 M0 for -23.4… and 100.7… without working
4.	Volume= $\pi\int_1^3 \sin^2 y\,\mathrm{d}y$	B1	Correct notation required
	$\pi\int_1^3 \frac{1-\cos 2y}{2}\,\mathrm{d}y$	M1	Integrable form with no more than 1 slip
	$\pi\left[\frac{1}{2}y - \frac{1}{4}\sin 2y\right]_1^3$	A1	oe cao
	$\pi\left[\left(\frac{3}{2} - \frac{1}{4}\sin 6\right) - \left(\frac{1}{2} - \frac{1}{4}\sin 2\right)\right]$	m1	Attempt to substitute in correct limits
	Volume = 4.08	A1	cao

5. a)	$\left(\begin{array}{ccc\|c} 1 & 2 & 0 & 3 \\ 2 & -5 & 3 & 8 \\ 0 & 6 & -2 & 0 \end{array}\right)$	M1	Attempt at row reduction
	$\left(\begin{array}{ccc\|c} 1 & 2 & 0 & 3 \\ 0 & -9 & 3 & 2 \\ 0 & 6 & -2 & 0 \end{array}\right)$	A1	1 row a multiple of another row
	$\left(\begin{array}{ccc\|c} 1 & 2 & 0 & 3 \\ 0 & -9 & 3 & 2 \\ 0 & 0 & 0 & \frac{4}{3} \end{array}\right)$	A1	oe
	Valid statement. Eg. As $0x + 0y + 0z \neq \frac{4}{3}$ there are no solutions.	E1	If M0, SC1 det A = 0 SC1 No **unique** solutions
b)	A correct statement involving 3 planes with no incorrect statements e.g. 3 planes do not meet at a single point	B1	FT their (a)
6.	$\cos 2\theta - \cos 4\theta = -2\sin\frac{2\theta + 4\theta}{2}\sin\frac{2\theta - 4\theta}{2}$	M1	M0 no working
	$-2\sin 3\theta \sin(-\theta) = \sin 3\theta$	A1	
	$2\sin 3\theta \sin\theta - \sin 3\theta = 0$		
	$\sin 3\theta\,(2\sin\theta - 1) = 0$	A1	
	$\sin 3\theta = 0 \qquad \sin\theta = \frac{1}{2}$	A1	FT one slip for A1A1A1 Both solutions
	$3\theta = 0, \pi, 2\pi, 3\pi$		
	$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi \qquad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$	A1A1	A1 each set of solutions If A1A1, penalise -1 for use of degrees
7. a)	$4x^2 + 10x - 24 = 4\left[x^2 + \frac{5}{2}x - 6\right]$	M1	$4x^2 + 10x - 24$ $= 4\left[x^2 + \frac{5}{2}x\right] - 24$
	$= 4\left[\left(x + \frac{5}{4}\right)^2 - \frac{121}{16}\right]$	m1	oe
	$= 4\left(x + \frac{5}{4}\right)^2 - \frac{121}{4}$	A1	
	Therefore, $a = 4 \quad b = \frac{5}{4} \quad c = -\frac{121}{4}$		

b)	METHOD 1:		
	$\int_3^5 \frac{6}{\sqrt{4x^2+10x-24}}\,\mathrm{d}x$		
	$= \int_3^5 \frac{6}{\sqrt{4\left(x+\frac{5}{4}\right)^2 - \frac{121}{4}}}\,\mathrm{d}x$	M1	M0 no working FT (a) for equivalent difficulty
	$= \int_3^5 \frac{6}{2\sqrt{\left(x+\frac{5}{4}\right)^2 - \frac{121}{16}}}\,\mathrm{d}x$	m1	Extracting a factor of $\sqrt{4}$ from denominator
	$= \left[3\cosh^{-1}\left(\frac{x+\frac{5}{4}}{\sqrt{\frac{121}{16}}}\right)\right]_3^5$	A1	oe
	$= \left[3\cosh^{-1}\left(\frac{4x+5}{11}\right)\right]_3^5$		
	$= \left[3\cosh^{-1}\left(\frac{25}{11}\right) - 3\cosh^{-1}\left(\frac{17}{11}\right)\right]$	m1	
	$= 1.379$	A1	cao Must be 3d.p.
	METHOD 2:		
	$\int_3^5 \frac{6}{\sqrt{4x^2+10x-24}}\,\mathrm{d}x$		
	$= \int_3^5 \frac{6}{\sqrt{4\left(x+\frac{5}{4}\right)^2 - \frac{121}{4}}}\,\mathrm{d}x$	(M1)	M0 no working FT (a) for equivalent difficulty
	$= \int_3^5 \frac{6}{2\sqrt{\left(x+\frac{5}{4}\right)^2 - \frac{121}{16}}}\,\mathrm{d}x$	(m1)	Extracting a factor of $\sqrt{4}$ from denominator
	$= \left[3\ln\left\{x+\frac{5}{4}+\sqrt{\left(x+\frac{5}{4}\right)^2 - \frac{121}{16}}\right\}\right]_3^5$	(A1)	
	$= 3\ln\left[\frac{25}{4}+\sqrt{\frac{504}{16}}\right] - 3\ln\left[\frac{17}{4}+\sqrt{\frac{168}{16}}\right]$	(m1)	
	$= 3\ln\left[\frac{25+\sqrt{504}}{17+\sqrt{168}}\right] = 3\ln\left[\frac{25+6\sqrt{14}}{17+2\sqrt{42}}\right]$		
	$= 1.379$	(A1)	cao Must be 3d.p.

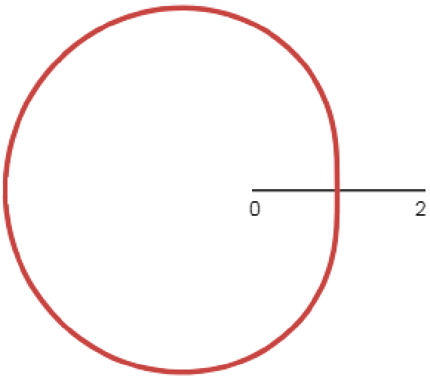
8.	$x = \sinh y$ $x = \dfrac{e^y - e^{-y}}{2}$ $2xe^y = (e^y)^2 - 1$ $\therefore\ (e^y)^2 - 2xe^y - 1 = 0$ Using quadratic formula, $e^y = \dfrac{2x \pm \sqrt{4x^2+4}}{2} \qquad \left(= x \pm \sqrt{x^2+1}\right)$ $y = \ln\left(x \pm \sqrt{x^2+1}\right)$ As $x - \sqrt{x^2+1} < 0$, $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$	B1 B1 M1 A1 A1 B1	 Allow omission of $\pm$ Justification may be seen earlier
9. a) i)	$\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)^3$ $\cos^3\frac{\theta}{3} + 3\cos^2\frac{\theta}{3}\left(i\sin\frac{\theta}{3}\right) + 3\cos\frac{\theta}{3}\left(i\sin\frac{\theta}{3}\right)^2 + \left(i\sin\frac{\theta}{3}\right)^3$ $= \cos^3\frac{\theta}{3} + 3i\cos^2\frac{\theta}{3}\sin\frac{\theta}{3} - 3\cos\frac{\theta}{3}\sin^2\frac{\theta}{3} - i\sin^3\frac{\theta}{3}$	 M1 A1	 Unsimplified Allow cis notation
ii)	$\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)^3 = \cos\theta + i\sin\theta$ $\therefore \cos\theta = \cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3}\sin^2\frac{\theta}{3}$ $= \cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3}\left(1 - \cos^2\frac{\theta}{3}\right)$ $= 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3}$	B1 M1 A1 A1	si FT (i) for sign error only cao convincing
b)	METHOD 1: $\dfrac{\cos\theta}{\cos\frac{\theta}{3}} = \dfrac{4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3}}{\cos\frac{\theta}{3}} = 1$ $4\cos^3\frac{\theta}{3} - 4\cos\frac{\theta}{3} = 0$ $4\cos\frac{\theta}{3}\left(\cos^2\frac{\theta}{3} - 1\right) = 0$ $\cos\frac{\theta}{3} = 0$ (not a possible solution in this equation) or $\cos\frac{\theta}{3} = \pm 1$ When $\cos\frac{\theta}{3} = 1,\ \ \frac{\theta}{3} = 2n\pi$ $\therefore\ \theta = 6n\pi$ When $\cos\frac{\theta}{3} = -1,\ \ \frac{\theta}{3} = \pi + 2n\pi$ $\therefore\ \theta = 3\pi + 6n\pi$ General solution: $\theta = 3n\pi$	M1 A1 A1 M1 A1 A1	Substitution Removing fraction All three (including ± 1) Use of general solution of $\cos\theta$ Either θ

	METHOD 2: $\frac{\cos\theta}{\cos\frac{\theta}{3}} = 1$ $\cos\theta - \cos\frac{\theta}{3} = 0$ Then, $-2\sin\frac{\theta+\frac{\theta}{3}}{2}\sin\frac{\theta-\frac{\theta}{3}}{2} = 0$ Therefore, $\sin\frac{2\theta}{3} = 0$ or $\sin\frac{\theta}{3} = 0$ $\frac{2\theta}{3} = n\pi$ or $\frac{\theta}{3} = n\pi$ $\theta = \frac{3}{2}n\pi$ or $\theta = 3n\pi$ Odd multiple of $\frac{3}{2}n\pi$ are not a solution because $\cos\theta = 0$ $\theta = 3n\pi$	 (B1) (M1) (A1) (A1) (M1) (A1)	 Both

10. a)	$\det A = 4(\lambda \times \lambda) - (8 \times 8)$ $\det A = 4\lambda^2 - 64$ Singular when det **A** = 0 METHOD 1: $4\lambda^2 - 64 = 0$ $\lambda^2 = 16$ $\lambda = \pm 4$ so there are two values where **A** is singular METHOD 2: $4\lambda^2 - 64 = 0$ Discriminant = $0^2 - (4 \times 4 \times -64) = 1024$ As 1024 > 0 there are two roots of the equation so there are two values where **A** is singular	M1 A1 M1 A1 (M1) (A1)	oe Must reference >0
b) i)	Cofactor matrix: $\begin{pmatrix} 9 & -8 & -12 \\ -24 & 12 & 32 \\ -16 & 8 & 12 \end{pmatrix}$ Adjugate matrix = $\begin{pmatrix} 9 & -24 & -16 \\ -8 & 12 & 8 \\ -12 & 32 & 12 \end{pmatrix}$	B3	All correct B2 for 7 or 8 correct B1 for 5 or 6 correct
ii)	$\det A = (4 \times 3^2) - 64 = -28$ $\therefore A^{-1} = \frac{1}{-28}\begin{pmatrix} 9 & -24 & -16 \\ -8 & 12 & 8 \\ -12 & 32 & 12 \end{pmatrix}$	B1 B1	FT their (a) FT their adjugate Mark final answer

11. a) i)	$y = e^{3x} \sin^{-1} x$ Use of product rule while differentiating $\dfrac{dy}{dx} = e^{3x}.\dfrac{1}{\sqrt{1-x^2}} + 3e^{3x} \sin^{-1} x$	M1 A2	 A1 each term ISW
ii)	METHOD 1: $y = \ln(\cosh(2x^2+7x))^2 = 2\ln(\cosh(2x^2+7x))$ $\dfrac{dy}{dx} = \dfrac{2 \times \sinh(2x^2+7x) \times (4x+7)}{\cosh(2x^2+7x)}$ METHOD 2: $y = \ln(\cosh(2x^2+7x))^2$ $\dfrac{dy}{dx} = \dfrac{2\cosh(2x^2+7x) \times \sinh(2x^2+7x) \times (4x+7)}{(\cosh(2x^2+7x))^2}$ $\dfrac{dy}{dx} = \dfrac{2 \times \sinh(2x^2+7x) \times (4x+7)}{\cosh(2x^2+7x)}$	M1 A1 A1 A1 (M1) (A1) (A1) (A1)	Log rule AND chain rule $\sinh(2x^2+7x)$ $4x+7$ oe Fully correct ISW Chain rule $\sinh(2x^2+7x)$ $4x+7$ oe Fully correct ISW
b)	METHOD 1: $1 = \dfrac{1}{\sqrt{1+(y^2)^2}} \times \left(2y\dfrac{dy}{dx}\right)$ $\sqrt{1+y^4} = 2y\dfrac{dy}{dx}$ $\dfrac{dy}{dx} = \dfrac{\sqrt{1+y^4}}{2y}$ METHOD 2: $y^2 = \sinh x$ $2y\dfrac{dy}{dx} = \cosh x$ $\dfrac{dy}{dx} = \dfrac{\cosh x}{2y}$ METHOD 3: $y = \pm\sqrt{\sinh x}$ $\dfrac{dy}{dx} = \pm\dfrac{1}{2}\sinh^{-\frac{1}{2}} x \cosh x$ THEN: When $x = 1$, $y = \pm 1.084$, $\dfrac{dy}{dx} = \pm 0.7117$ $y - 1.084 = 0.7117(x-1)$ $y + 1.084 = -0.7117(x-1)$	M1 A1 A1 A1 (M1) (A1) (A1) (A1) (M1) (A1) (A1) (A1) B1 A1 B1 B1	Must see chain rule Differentiate $\sinh^{-1}$ $2y\dfrac{dy}{dx}$ $2y\dfrac{dy}{dx}$ Cosh $\frac{1}{2}\sinh^{-\frac{1}{2}} x$ Cosh $\pm$ Both cao Both FT their y and dy/dx FT their y and dy/dx

12.	Solve auxiliary $3t^2 + 5t - 2 = 0$ $(3t-1)(t+2) = 0$ $t = \frac{1}{3}$ or $t = -2$	M1 A1	M0A0 no working Both values
	Complementary function: $y = Ae^{\frac{1}{3}x} + Be^{-2x}$		
	Use particular integral of the form $Cx^2 + Dx + E$ $\frac{dy}{dx} = 2Cx + D$ $\frac{d^2y}{dx^2} = 2C$	M1 A1	 Both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$
	Therefore, $6C + 5(2Cx + D) - 2(Cx^2 + Dx + E) = 8 + 6x - 2x^2$	A1	Substitution
	$-2C = -2 \rightarrow C = 1$ $10C - 2D = 6 \rightarrow D = 2$ $6C + 5D - 2E = 8 \rightarrow E = 4$	A1	All values
	General Solution: $y = Ae^{\frac{1}{3}x} + Be^{-2x} + x^2 + 2x + 4$	M1	FT C,D,E for M1A1M1A1 Sub and differentiate
	$\frac{dy}{dx} = \frac{1}{3}Ae^{\frac{1}{3}x} - 2Be^{-2x} + 2x + 2$	A1	
	When $x = 0$, $y = A + B + 4 = 6$	M1	Substitution
	$\frac{dy}{dx} = \frac{1}{3}A - 2B + 2 = 5$	A1	Both y and $\frac{dy}{dx}$
	Solving, $A = 3$ and $B = -1$	B1	cao
	Therefore, $y = 3e^{\frac{1}{3}x} - e^{-2x} + x^2 + 2x + 4$	B1	cao

13. a)		G1 G1	For **shape**, to include reflection in the initial line. Fully correct
b) i)	$y = r\sin\theta$ $y = (2 - \cos\theta)\sin\theta$ $y = 2\sin\theta - \sin\theta\cos\theta$ THEN $\left(y = 2\sin\theta - \frac{1}{2}\sin 2\theta\right)$ $\frac{dy}{d\theta} = 2\cos\theta - \cos 2\theta$ When parallel to initial line, $2\cos\theta - \cos 2\theta = 0$ $2\cos\theta - (2\cos^2\theta - 1) = 0$ $2\cos^2\theta - 2\cos\theta - 1 = 0$ OR $\frac{dy}{d\theta} = 2\cos\theta - (\cos^2\theta - \sin^2\theta)$ When parallel to initial line, $2\cos\theta - (\cos^2\theta - \sin^2\theta) = 0$ $2\cos\theta - \cos^2\theta + (1 - \cos^2\theta) = 0$ $2\cos^2\theta - 2\cos\theta - 1 = 0$	M1 A1 m1 A1 (A1) (m1) (A1)	 convincing convincing
ii)	Solving $\cos\theta = \frac{2 \pm \sqrt{4+8}}{4}$ $\cos\theta = 1.366$ therefore no solutions or $\cos\theta = -0.366$ $\therefore\ \theta = 1.9455 \text{ or } 4.3377$ $r = 2.366$	M1 A1 A1 A1 B1	 Both values FT their θ

14.	$\dfrac{6x^2+2x+16}{x^3-x^2+3x-3}=\dfrac{6x^2+2x+16}{(x-1)(x^2+3)}$	M1 A1	Linear × Quadratic
	$\dfrac{6x^2+2x+16}{(x-1)(x^2+3)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+3}$	M1	FT their factorising if linear × quadratic of equivalent difficulty
	$6x^2+2x+16=A(x^2+3)+(Bx+C)(x-1)$	A1	
	When $x=1$, $24=4A$ $\rightarrow A=6$		
	When $x=0$, $16=3A-C$ $\rightarrow C=2$		
	Compare coefficients of x^2: $6=A+B$ $\therefore B=0$	A2	A2 all 3 values A1 any 2 values If M0, SC1 for A = 6, B = 0, C = 2.
	$\int_2^4 \dfrac{6x^2+2x+16}{x^3-x^2+3x-3}\,\mathrm{d}x$ $=\int_2^4\left(\dfrac{6}{x-1}+\dfrac{2}{x^2+3}\right)\mathrm{d}x$	M1	FT their A, B, C provided $a\neq 0$ and $c\neq 0$
	$=\left[6\ln(x-1)+\dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{x}{\sqrt{3}}\right]_2^4$	A2	A1 each term
	$=7.93362-0.98966=6.944$	A1	cao Answer only 0 marks